COMP331/COMP557: Optimisation

Martin Gairing

Computer Science Department

University of Liverpool

1st Semester 2018/19

Material adapted from a course by Martin Skutella at TU Berlin

My Background

FH Esslingen

- 1995-2000: Diplom (Electrical Engineering)

Clemson University

- 2000-2001: MSc (Computer Science)

University of Paderborn

- 2002-2007: PhD + Postdoc

International Computer Science Institute Berkeley

- 2007- 2009: Postdoc

Liverpool University

- Since 2009: Lecturer/Senior Lecturer

Administrative Details

Lectures:

- Mondays, 11:00 - 12:00
- Tuesdays, 10:00 - 11:00
- Thursdays, 12:00 -13:00

Tutorials:

- Flávia Alves (F.Alves@liverpool.ac.uk)
- starting from Friday 28 September

Assessment:

- 25 % continuous assessment
- 75 % final exam

Course Aims

- To provide a foundation for modelling various continuous and discrete optimisation problems.
- To provide the tools and paradigms for the design and analysis of algorithms for continuous and discrete optimisation problems. Apply these tools to real-world problems.
- To review the links and interconnections between optimisation and computational complexity theory.
- To provide an in-depth, systematic and critical understanding of selected significant topics at the intersection of optimisation, algorithms and (to a lesser extent) complexity theory, together with the related research issues.

Learning Outcomes

Upon completion of the module you should have:

- A critical awareness of current problems and research issues in the field of optimisation.
- The ability to formulate optimisation models for the purpose of modelling particular applications.
- The ability to use appropriate algorithmic paradigms and techniques in context of a particular optimisation model.
- The ability to read, understand and communicate research literature in the field of optimisation.
- The ability to recognise potential research opportunities and research directions.

Outline

Chapter 1: Introduction

A Motivating (and Refreshing) Example

Small brewery produces ale and beer.

- Production limited by scarce resources: corn, hops, barley malt.
- Recipes for ale and beer require different proportions of resources.

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Is this best possible?

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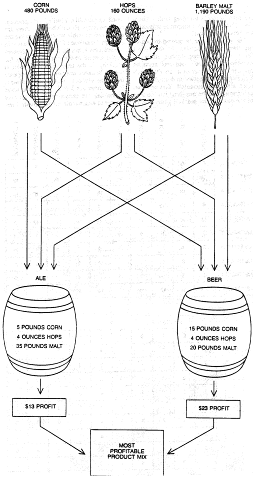

- Mathematical Formulation:

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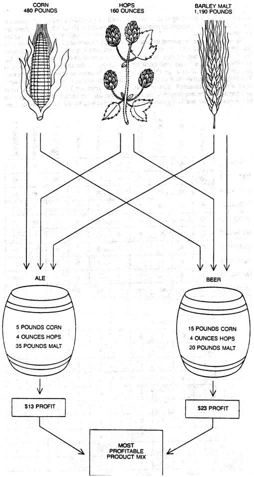

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$$\max \quad 13\,A \quad + \quad 23\,B \qquad \text{Profit}$$

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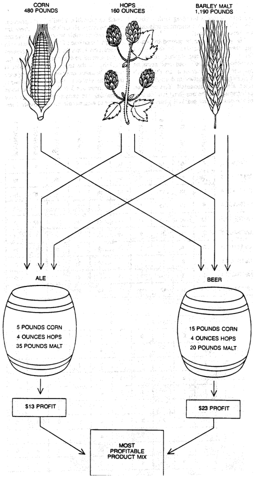

- Mathematical Formulation:

$$\begin{array}{lrcrcll} \max & 13\,A & + & 23\,B & & & \text{Profit} \\ \text{s.t.} & 5\,A & + & 15\,B & \leq 480 & & \text{Corn} \end{array}$$

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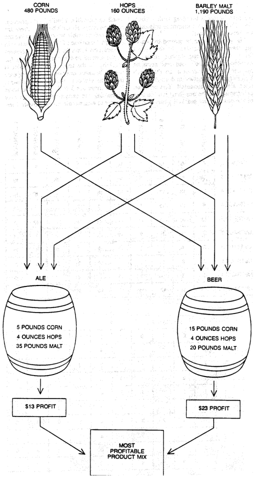

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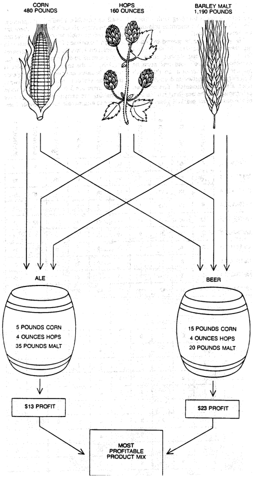

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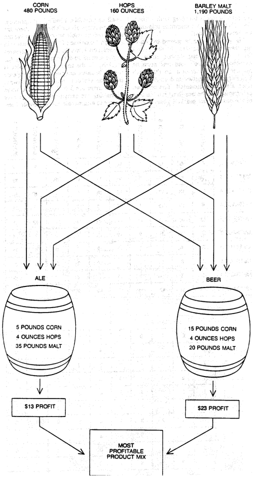

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 & & & A, B & \geq 0 &
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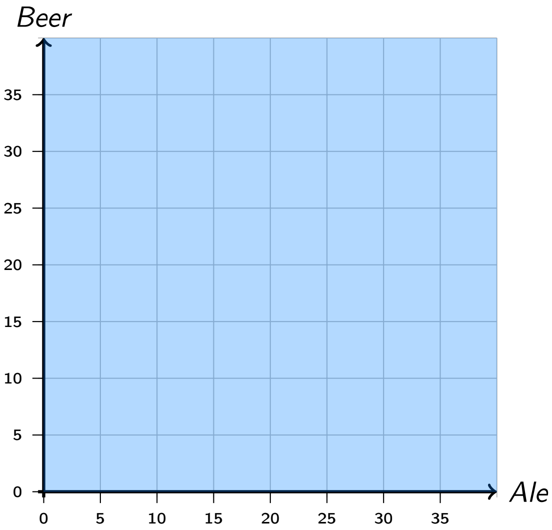

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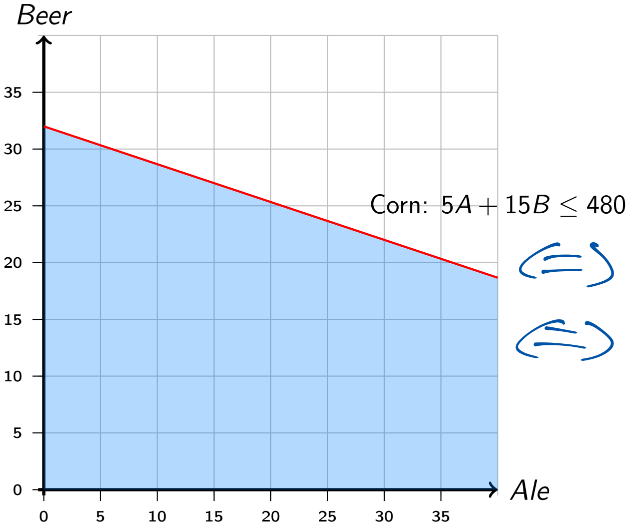

$$\iff 15B \leq 480 - 5A$$

$$\iff B \leq 32 - \frac{1}{3}A$$

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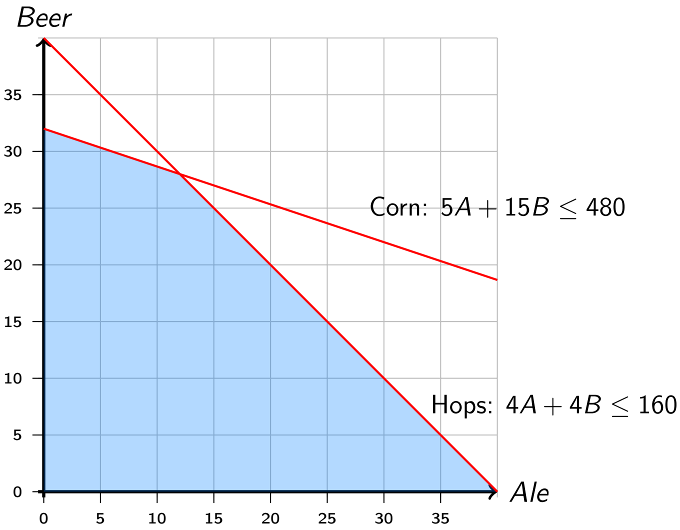

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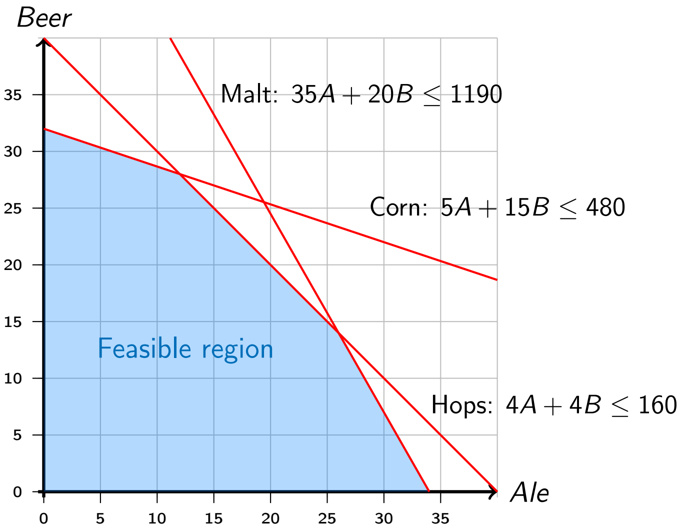

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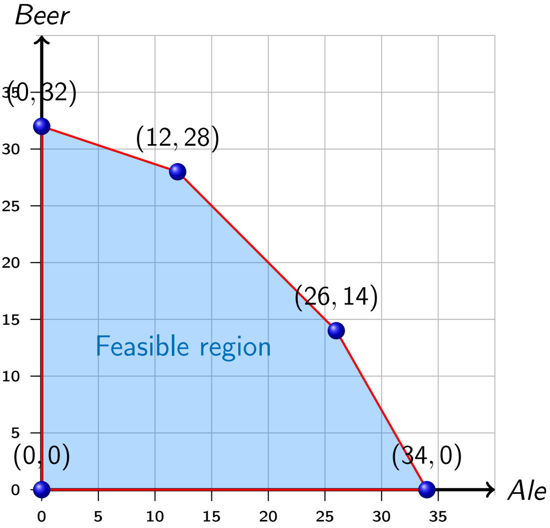

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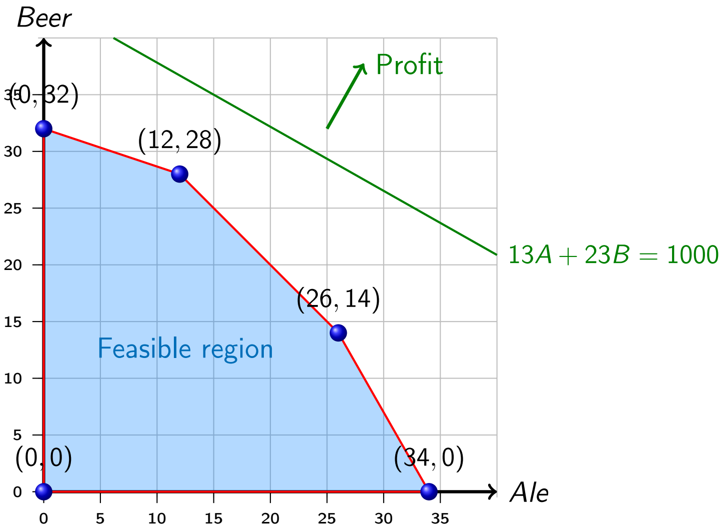

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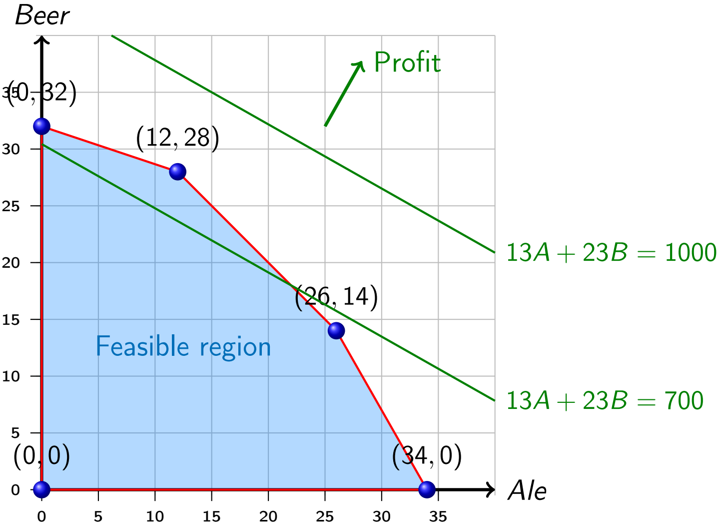

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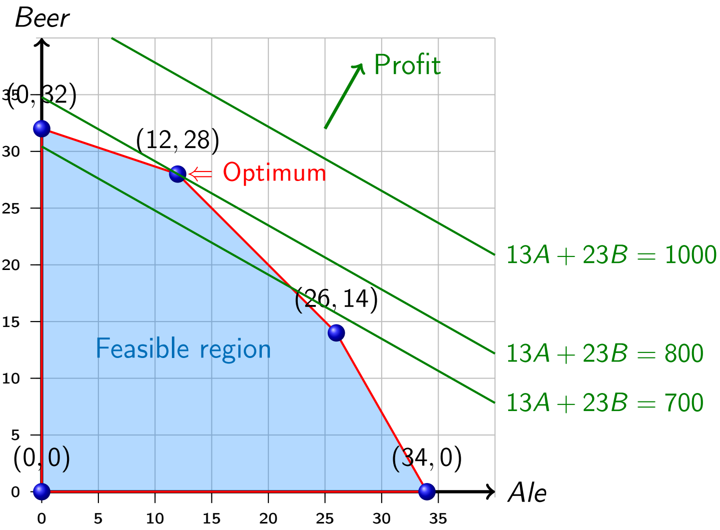

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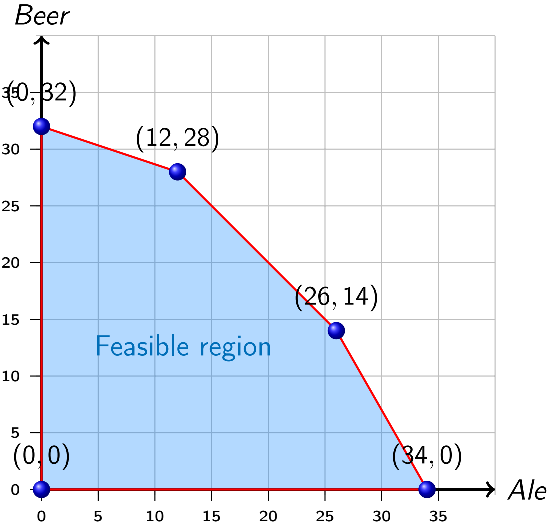

Observation: Regardless of objective function coefficients, an optimal solution occurs at an extreme point (vertex).

Terminology and Notation

Numbers:

- $\mathbb{R}$ ... set of real numbers
- $\mathbb{R}_{\geq 0}$ or $\mathbb{R}_+$ ... set of non-negative real numbers
- $\mathbb{R}^n$ ... n-dimensional real vector space

- $\mathbb{Z}$, $\mathbb{Z}_{\geq 0}$, $\mathbb{Z}^n$ ... set of integers, non-negative integers, n-dimensional ...

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Sets:

- $S = \{s_1, s_2, \cdots, s_k\}$ . . . a set of k elements
- $S = \{x \mid P(x)\}$ . . . set of elements x for which condition P is true
  - Example: $\mathbb{Z}_{\geq 0} = \{i \mid i \in \mathbb{Z} \text{ and } i \geq 0\}$
- $|S|$ . . . size (number of elements) of a finite set S
- 2^S . . . set of all subsets of S
  - e.g.: $2^{\{a,b,c\}} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$
- $\mu : S \mapsto T$ . . . μ is a mapping (or function) from set S to set T

Terminology and Notation – Linear Algebra

- matrix of dimension $m \times n$:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} = \begin{pmatrix} | & | & & | \\ A_1 & A_2 & \dots & A_n \\ | & | & & | \end{pmatrix} = \begin{pmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{pmatrix}$$

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- Inner product of $x, y \in \mathbb{R}^n$: $\quad x^T y = y^T x = \sum_{i=1}^n x_i y_i$

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- Inner product of $x, y \in \mathbb{R}^n$: $x^T y = y^T x = \sum_{i=1}^n x_i y_i$
- Matrix equation $Ax = b$
  is equivalent to $a_i^T x = b_i$ for all $i \in \{1, \dots, m\}$
  (b is an m-vector, b_i is its i'th component)

Terminology and Notation – Linear Algebra

- $\det(A)$ … determinant of a matrix
  - e.g.: $\det\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$

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- e_i . . . unit vector (dimension from context)
  - 1 in i'th component, 0 else
  - e.g. (dimension 3): $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

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- $I = \begin{pmatrix} | & | & & | \\ e_1 & e_2 & \cdots & e_n \\ | & | & & | \end{pmatrix}$ . . . identity matrix (dimension from context, here n)
  - 1 on main diagonal, 0 else
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- $\text{rank}(A)$ = size of the largest set of linearly independent columns
  = size of the largest set of linearly independent rows
- A^{-1} … matrix inverse of square matrix A
  - $A^{-1}A = A\,A^{-1} = I$
  - A^{-1} exists if and only if $\det(A) \neq 0$