Optimization Problems

Generic optimization problem

Given: set X, function $f : X \to \mathbb{R}$
Task: find $x^* \in X$ maximizing (minimizing) $f(x^*)$, i. e.,

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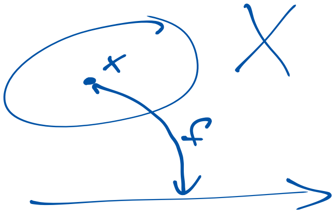

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Problem: Too general to say anything meaningful!

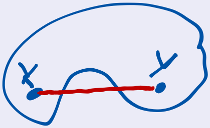
Convex Optimization Problems

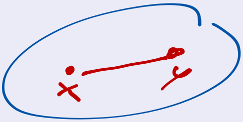
Definition 1.1.

Let $X \subseteq \mathbb{R}^n$ and $f : X \to \mathbb{R}$.

- a X is convex if for all $x, y \in X$ and $0 \leq \lambda \leq 1$ it holds that

$$\lambda \cdot x + (1 - \lambda) \cdot y \in X \quad .$$





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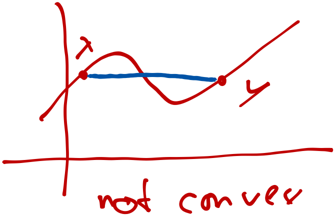

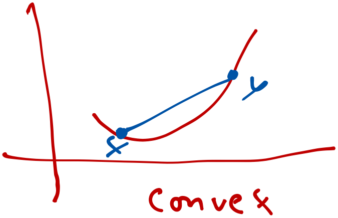

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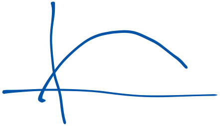
c. If X and f are both convex, then $\min\{f(x) \mid x \in X\}$ is a convex optimization problem.

Note: $f : X \mapsto \mathbb{R}$ is called concave if $-f$ is convex.



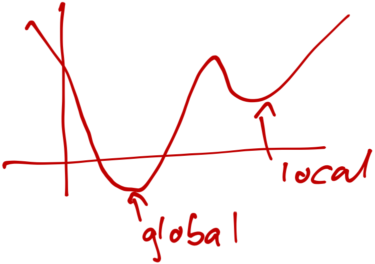
Local and Global Optimality

Definition 1.2.

Let $X \subseteq \mathbb{R}^n$ and $f : X \mapsto \mathbb{R}$.
$x' \in X$ is a local optimum of the optimization problem $\min\{f(x) \mid x \in X\}$ if there is an $\varepsilon > 0$ such that

$$f(x') \leq f(x) \qquad \text{for all } x \in X \text{ with } \|x' - x\|_2 \leq \varepsilon.$$

distance between x and x'



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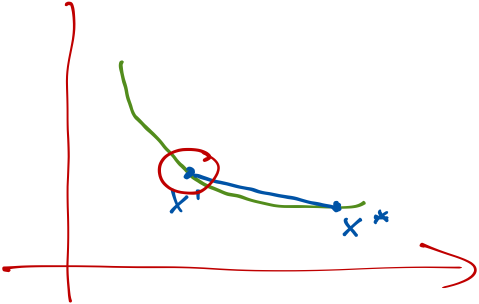
Theorem 1.3.

For a convex optimization problem, every local optimum is a (global) optimum.

Proof by contradiction:

Assume x' is local optimum and x^* is global optimum and

$$f(x^*) < f(x').$$



$\Rightarrow$ contradicts local optimality of x'

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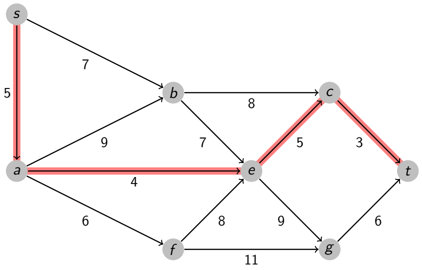
Example: Shortest Path Problem



Given: directed graph $D = (V, A)$,
weight function $w : A \to \mathbb{R}_{\geq 0}$,
start node $s \in V$,
destination node $t \in V$.

Task: find s-t-path of minimum weight.

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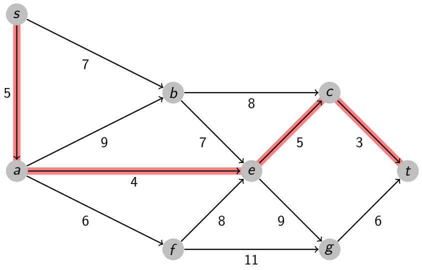

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That is, $X = \{P \subseteq A \mid P \text{ is } s\text{-}t\text{-path in } D\}$ and $f : X \to \mathbb{R}$ is given by

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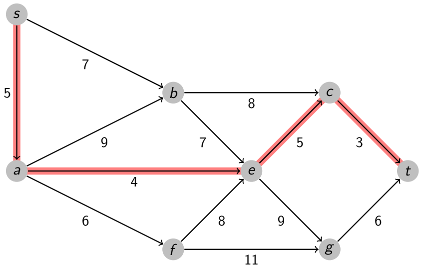

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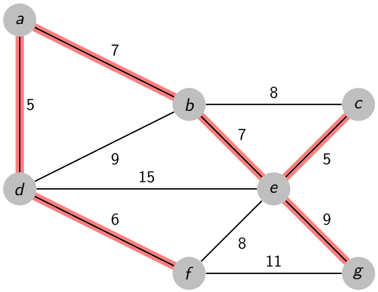
Remark.

Note that the finite set of feasible solutions X is only implicitly given by D.
This holds for all interesting problems in combinatorial optimization!

Example: Minimum Spanning Tree (MST) Problem



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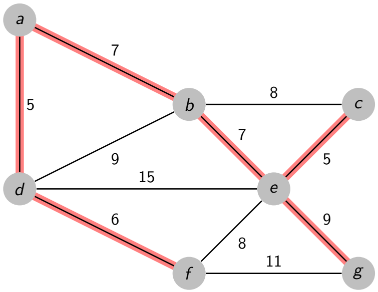

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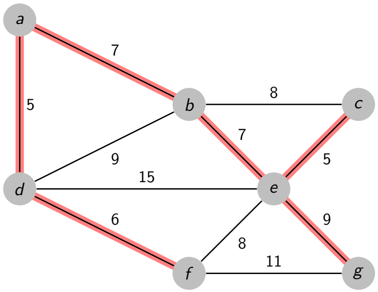

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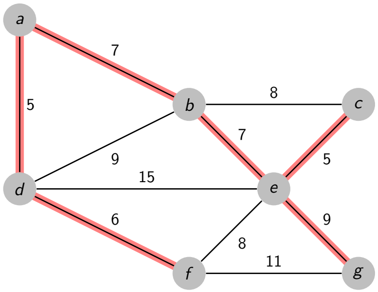

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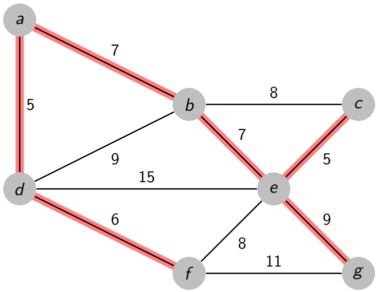

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Example: Minimum Cost Flow Problem

Given: directed graph $D = (V, A)$, with arc *capacities* $u : A \to \mathbb{R}_{\geq 0}$, arc costs $c : A \to \mathbb{R}$, and node *balances* $b : V \to \mathbb{R}$.

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- nodes $v \in V$ with $b(v) > 0$ ($b(v) < 0$) have *supply* (*demand*) and are called *sources* (*sinks*)
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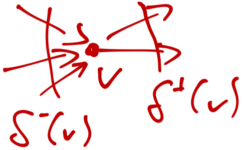
Task: find a *flow* $x : A \to \mathbb{R}_{\geq 0}$ obeying capacities and satisfying all supplies and demands, that is,

$$\begin{aligned} 0 \leq x(a) \leq u(a) & \qquad \text{for all } a \in A, \\ \sum_{a \in \delta^+(v)} x(a) - \sum_{a \in \delta^-(v)} x(a) = b(v) & \qquad \text{for all } v \in V, \end{aligned}$$

such that x has minimum cost $c(x) := \sum_{a \in A} c(a) \cdot x(a)$.



Example: Minimum Cost Flow Problem (cont.)

Formulation as a linear program (LP):

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- Notice that (1.1) is a linear function of x and (1.2) – (1.4) are linear equations and linear inequalities, respectively. $\longrightarrow$ linear program