General Linear Program

$$
\begin{array}{lll}
\text{minimize} & c^T \cdot x = c_1 x_1 + \ldots + c_n x_n & \\
\text{subject to} & {a_i}^T \cdot x \geq b_i & \text{for } i \in M_1, \quad (2.1)\\
& {a_i}^T \cdot x = b_i & \text{for } i \in M_2, \quad (2.2)\\
& {a_i}^T \cdot x \leq b_i & \text{for } i \in M_3, \quad (2.3)\\
& x_j \geq 0 & \text{for } j \in N_1, \quad (2.4)\\
& x_j \leq 0 & \text{for } j \in N_2, \quad (2.5)
\end{array}
$$

with $c \in \mathbb{R}^n$, $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$ for $i \in M_1 \dot\cup M_2 \dot\cup M_3$ (finite index sets), and $N_1, N_2 \subseteq \{1, \ldots, n\}$ given.

- $x \in \mathbb{R}^n$ satisfying constraints (2.1) – (2.5) is a feasible solution.
- feasible solution x^* is optimal solution if

$$c^T \cdot x^* \leq c^T \cdot x \qquad \text{for all feasible solutions } x.$$

- linear program is unbounded if, for all $k \in \mathbb{R}$, there is a feasible solution $x \in \mathbb{R}^n$ with $c^T \cdot x \leq k$.

Special Forms of Linear Programs

- maximizing $c^T \cdot x$ is equivalent to minimizing $(-c)^T \cdot x$.

Special Forms of Linear Programs

- maximizing $c^T \cdot x$ is equivalent to minimizing $(-c)^T \cdot x$.
- any linear program can be written in the form

$$\begin{aligned} \text{minimize} \quad & c^T \cdot x \\ \text{subject to} \quad & A \cdot x \geq b \end{aligned}$$

for some $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$:

Special Forms of Linear Programs

- maximizing $c^T \cdot x$ is equivalent to minimizing $(-c)^T \cdot x$.
- any linear program can be written in the form

$$\begin{aligned} \text{minimize} \quad & c^T \cdot x \\ \text{subject to} \quad & A \cdot x \geq b \end{aligned}$$

  for some $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$:
  - rewrite $a_i{}^T \cdot x = b_i$ as: $a_i{}^T \cdot x \geq b_i \ \wedge \ a_i{}^T \cdot x \leq b_i$,

Special Forms of Linear Programs

- maximizing $c^T \cdot x$ is equivalent to minimizing $(-c)^T \cdot x$.
- any linear program can be written in the form

$$\begin{aligned} \text{minimize} \quad & c^T \cdot x \\ \text{subject to} \quad & A \cdot x \geq b \end{aligned}$$

  for some $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$:

  - rewrite $a_i{}^T \cdot x = b_i$ as: $a_i{}^T \cdot x \geq b_i \quad \wedge \quad a_i{}^T \cdot x \leq b_i$,
  - rewrite $a_i{}^T \cdot x \leq b_i$ as: $(-a_i)^T \cdot x \geq -b_i$.

Special Forms of Linear Programs

- maximizing $c^T \cdot x$ is equivalent to minimizing $(-c)^T \cdot x$.
- any linear program can be written in the form

$$\begin{aligned} \text{minimize} \quad & c^T \cdot x \\ \text{subject to} \quad & A \cdot x \geq b \end{aligned}$$

  for some $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$:

  - rewrite $a_i{}^T \cdot x = b_i$ as: $a_i{}^T \cdot x \geq b_i \quad \wedge \quad a_i{}^T \cdot x \leq b_i$,
  - rewrite $a_i{}^T \cdot x \leq b_i$ as: $(-a_i)^T \cdot x \geq -b_i$.

- Linear program in standard form:

$$\begin{aligned} \min \quad & c^T \cdot x \\ \text{s.t.} \quad & A \cdot x = b \\ & x \geq 0 \end{aligned}$$

  with $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$.

Example: Diet Problem

Given:

- n different foods, m different nutrients
- $a_{ij} :=$ amount of nutrient i in one unit of food j
- $b_i :=$ requirement of nutrient i in some ideal diet
- $c_j :=$ cost of one unit of food j

Example: Diet Problem

Given:

- n different foods, m different nutrients
- $a_{ij} :=$ amount of nutrient i in one unit of food j
- $b_i :=$ requirement of nutrient i in some ideal diet
- $c_j :=$ cost of one unit of food j

Task: find a cheapest ideal diet consisting of foods $1, \ldots, n$.

Example: Diet Problem

Given:

- n different foods, m different nutrients
- $a_{ij} :=$ amount of nutrient i in one unit of food j
- $b_i :=$ requirement of nutrient i in some ideal diet
- $c_j :=$ cost of one unit of food j

Task: find a cheapest ideal diet consisting of foods $1, \ldots, n$.

LP formulation: Let $x_j :=$ number of units of food j in the diet:

$$\begin{aligned} \min \quad & c^T \cdot x \\ \text{s.t.} \quad & A \cdot x = b \\ & x \geq 0 \end{aligned} \qquad \text{or} \qquad \begin{aligned} \min \quad & c^T \cdot x \\ \text{s.t.} \quad & A \cdot x \geq b \\ & x \geq 0 \end{aligned}$$

with $A = (a_{ij}) \in \mathbb{R}^{m \times n}$, $b = (b_i) \in \mathbb{R}^m$, $c = (c_j) \in \mathbb{R}^n$.

→ standard form

Reduction to Standard Form

Any linear program can be brought into standard form:

- elimination of free (unbounded) variables x_j:

  replace x_j with $x_j^+, x_j^- \geq 0$: $\quad x_j = x_j^+ - x_j^-$

- elimination of non-positive variables x_j:

  replace $x_j \leq 0$ with $(-x_j) \geq 0$.

- elimination of inequality constraint ${a_i}^T \cdot x \leq b_i$:

  introduce slack variable $s \geq 0$ and rewrite: ${a_i}^T \cdot x + s = b_i$

- elimination of inequality constraint ${a_i}^T \cdot x \geq b_i$:

  introduce slack variable $s \geq 0$ and rewrite: ${a_i}^T \cdot x - s = b_i$

Example

The linear program

$x_2 \mapsto x_2^+ - x_2^-$

$$\begin{array}{lrcrl} \min & 2\,x_1 & + & 4\,x_2 & \\ \text{s.t.} & x_1 & + & x_2 & \geq 3 \\ & 3\,x_1 & + & 2\,x_2 & = 14 \\ & x_1 & & & \geq 0 \end{array}$$

$\Rightarrow x_1 + x_2 - x_3 = 3$

$x_2^+ \geq 0$

$x_2^- \geq 0$

$x_3 \geq 0$

Example

The linear program

$$\begin{array}{rrcrl} \min & 2\,x_1 & + & 4\,x_2 & \\ \text{s.t.} & x_1 & + & x_2 & \geq 3 \\ & 3\,x_1 & + & 2\,x_2 & = 14 \\ & x_1 & & & \geq 0 \end{array}$$

is equivalent to the standard form problem

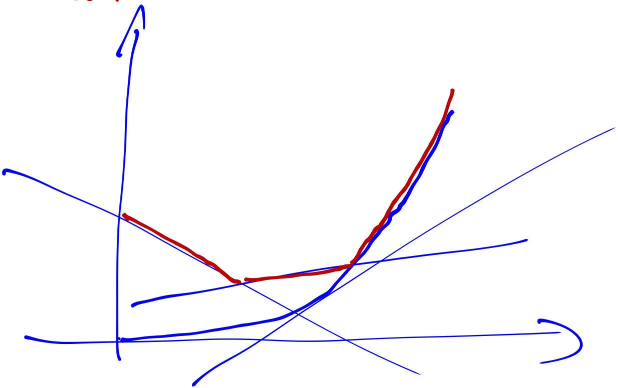
$$\begin{array}{rrcrcrcrl} \min & 2\,x_1 & + & 4\,x_2^+ & - & 4\,x_2^- & & & \\ \text{s.t.} & x_1 & + & x_2^+ & - & x_2^- & - & x_3 & = 3 \\ & 3\,x_1 & + & 2\,x_2^+ & - & 2\,x_2^- & & & = 14 \\ & & & & & & & x_1,\, x_2^+,\, x_2^-,\, x_3 & \geq 0 \end{array}$$

Affine Linear and Convex Functions

Lemma 2.1.

a. An affine linear function $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(x) = c^T \cdot x + d$ with $c \in \mathbb{R}^n$, $d \in \mathbb{R}$, is both convex and concave.

b. If $f_1, \ldots, f_k : \mathbb{R}^n \to \mathbb{R}$ are convex functions, then $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(x) := \max_{i=1,\ldots,k} f_i(x)$ is also convex.



Piecewise Linear Convex Objective Functions

Let $c_1, \ldots, c_k \in \mathbb{R}^n$ and $d_1, \ldots, d_k \in \mathbb{R}$.

Consider piecewise linear convex function: $x \mapsto \max_{i=1,\ldots,k} {c_i}^T \cdot x + d_i$:

Piecewise Linear Convex Objective Functions

Let $c_1, \ldots, c_k \in \mathbb{R}^n$ and $d_1, \ldots, d_k \in \mathbb{R}$.

Consider piecewise linear convex function: $x \mapsto \max_{i=1,\ldots,k} {c_i}^T \cdot x + d_i$:

$$
\begin{aligned}
\min \quad & \max_{i=1,\ldots,k} {c_i}^T \cdot x + d_i \\
\text{s.t.} \quad & A \cdot x \geq b
\end{aligned}
$$

Piecewise Linear Convex Objective Functions

Let $c_1, \dots, c_k \in \mathbb{R}^n$ and $d_1, \dots, d_k \in \mathbb{R}$.

Consider piecewise linear convex function: $x \mapsto \max_{i=1,\dots,k} {c_i}^T \cdot x + d_i$:

$$\begin{aligned} \min \quad & \max_{i=1,\dots,k} {c_i}^T \cdot x + d_i \\ \text{s.t.} \quad & A \cdot x \geq b \end{aligned} \qquad \longleftrightarrow \qquad \min \quad z$$

Piecewise Linear Convex Objective Functions

Let $c_1, \ldots, c_k \in \mathbb{R}^n$ and $d_1, \ldots, d_k \in \mathbb{R}$.

Consider piecewise linear convex function: $x \mapsto \max_{i=1,\ldots,k} {c_i}^T \cdot x + d_i$:

$$\begin{array}{ll} \min & \max\limits_{i=1,\ldots,k} {c_i}^T \cdot x + d_i \\ \text{s.t.} & A \cdot x \geq b \end{array} \qquad \longleftrightarrow \qquad \begin{array}{lll} \min & z & \\ \text{s.t.} & z \geq {c_i}^T \cdot x + d_i & \text{for all } i \end{array}$$

Piecewise Linear Convex Objective Functions

Let $c_1, \ldots, c_k \in \mathbb{R}^n$ and $d_1, \ldots, d_k \in \mathbb{R}$.

Consider piecewise linear convex function: $x \mapsto \max_{i=1,\ldots,k} {c_i}^T \cdot x + d_i$:

$$\begin{aligned} \min \quad & \max_{i=1,\ldots,k} {c_i}^T \cdot x + d_i \\ \text{s.t.} \quad & A \cdot x \geq b \end{aligned} \qquad \longleftrightarrow \qquad \begin{aligned} \min \quad & z \\ \text{s.t.} \quad & z \geq {c_i}^T \cdot x + d_i \qquad \text{for all } i \\ & A \cdot x \geq b \end{aligned}$$

Piecewise Linear Convex Objective Functions

Let $c_1, \ldots, c_k \in \mathbb{R}^n$ and $d_1, \ldots, d_k \in \mathbb{R}$.

Consider piecewise linear convex function: $x \mapsto \max_{i=1,\ldots,k} {c_i}^T \cdot x + d_i$:

$$\begin{array}{ll} \min & \max\limits_{i=1,\ldots,k} {c_i}^T \cdot x + d_i \\ \text{s.t.} & A \cdot x \geq b \end{array} \quad \longleftrightarrow \quad \begin{array}{lll} \min & z & \\ \text{s.t.} & z \geq {c_i}^T \cdot x + d_i & \text{for all } i \\ & A \cdot x \geq b & \end{array}$$

Example: let $c_1, \ldots, c_n \geq 0$

$$\begin{array}{ll} \min & \sum_{i=1}^{n} c_i \cdot |x_i| \\ \text{s.t.} & A \cdot x \geq b \end{array}$$

$|x_i|$ is smallest z_i satifying

$z_i \geq x_i$

and $z_i \geq -x_i$

Piecewise Linear Convex Objective Functions

Let $c_1, \dots, c_k \in \mathbb{R}^n$ and $d_1, \dots, d_k \in \mathbb{R}$.

Consider piecewise linear convex function: $x \mapsto \max_{i=1,\dots,k} {c_i}^T \cdot x + d_i$:

$$\begin{aligned} \min \quad & \max_{i=1,\dots,k} {c_i}^T \cdot x + d_i \\ \text{s.t.} \quad & A \cdot x \geq b \end{aligned} \qquad \longleftrightarrow \qquad \begin{aligned} \min \quad & z \\ \text{s.t.} \quad & z \geq {c_i}^T \cdot x + d_i \qquad \text{for all } i \\ & A \cdot x \geq b \end{aligned}$$

Example: let $c_1, \dots, c_n \geq 0$

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} c_i \cdot |x_i| \\ \text{s.t.} \quad & A \cdot x \geq b \end{aligned} \qquad \leftrightarrow \qquad \begin{aligned} \min \quad & \sum_{i=1}^{n} c_i \cdot z_i \\ \text{s.t.} \quad & z_i \geq x_i \quad \forall i \\ & z_i \geq -x_i \quad \forall i \\ & A \cdot x \geq b \end{aligned}$$

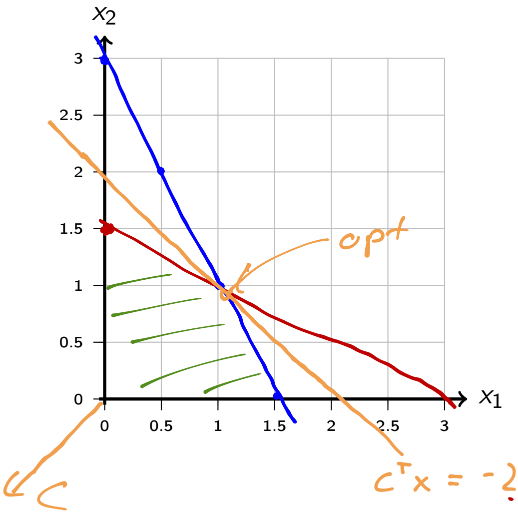
Graphical Representation and Solution

2D example:

$$
\begin{array}{llcrl}
\min & -x_1 & - & x_2 & \\
\text{s.t.} & x_1 & + & 2\,x_2 & \leq 3 \\
 & 2\,x_1 & + & x_2 & \leq 3 \\
 & & & x_1,\ x_2 & \geq 0
\end{array}
$$

$$\Leftrightarrow \quad x_2 \leq \frac{3}{2} - \frac{1}{2} x_1$$

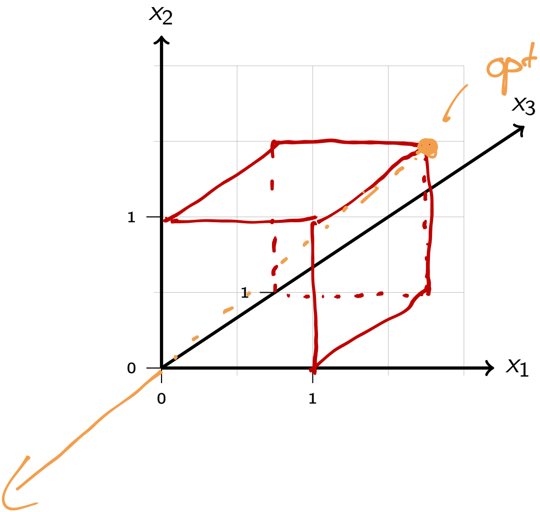
$$\Leftrightarrow \quad x_2 \leq 3 - 2 x_1$$



Graphical Representation and Solution (cont.)

3D example:

$$\begin{aligned} \min \quad & -x_1 - x_2 - x_3 \\ \text{s.t.} \quad & x_1 \leq 1 \\ & x_2 \leq 1 \\ & x_3 \leq 1 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$



Graphical Representation and Solution (cont.)

another 2D example:

$$\begin{array}{lrcrl} \min & c_1\, x_1 & + & c_2\, x_2 & \\ \text{s.t.} & -x_1 & + & x_2 & \leq 1 \\ & & & x_1,\, x_2 & \geq 0 \end{array}$$