Graphical Representation and Solution (cont.)

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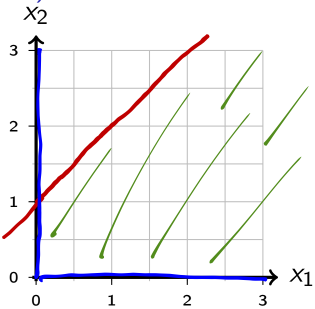

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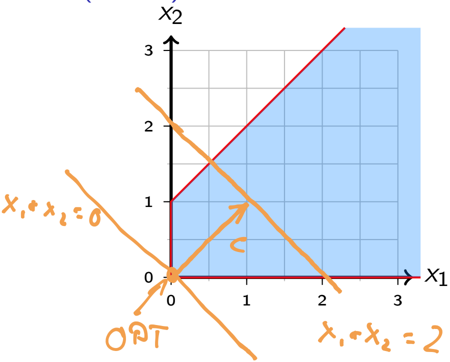

- for $c = (1,1)^T$

$x_1 + x_2 = 2$

$\Leftrightarrow \quad x_2 = 2 - x_1$

Graphical Representation and Solution (cont.)

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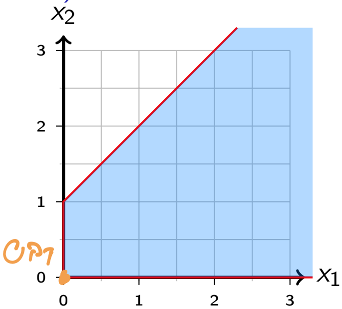

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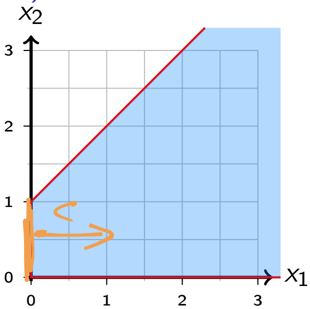

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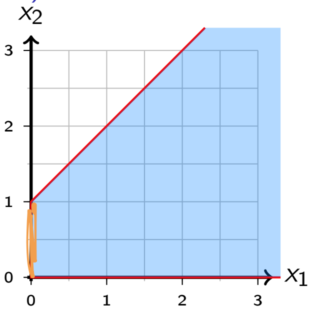

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- for $c = (1,0)^T$, the optimal solutions are exactly the points

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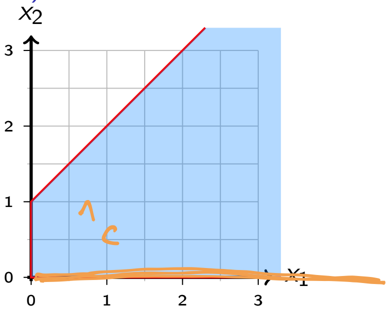

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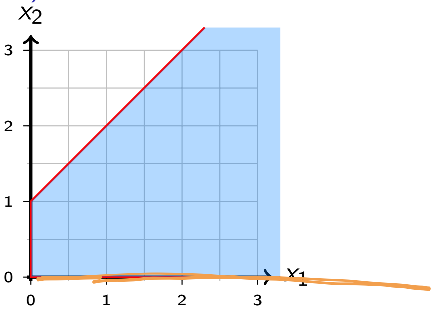

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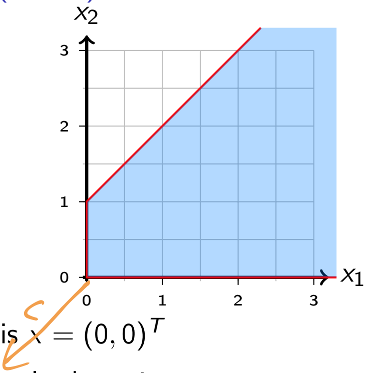

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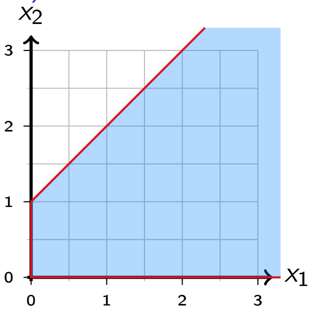

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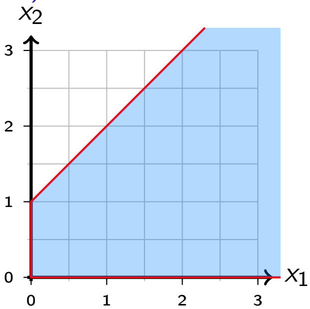

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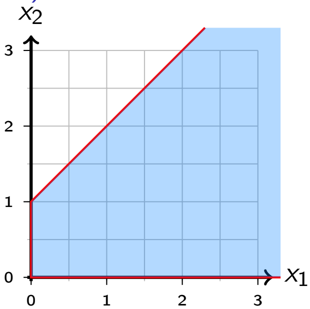

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Properties of the Set of Optimal Solutions

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These are indeed all cases that can occur in general (see later).

Visualizing LPs in Standard Form

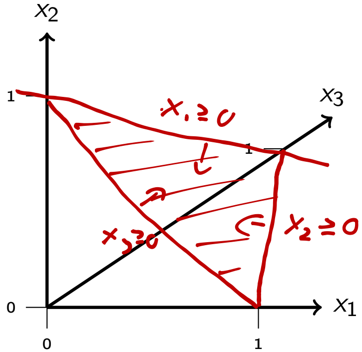
Example:

Let $A = (1,1,1) \in \mathbb{R}^{1\times 3}$, $b = (1) \in \mathbb{R}^1$ and consider the set of feasible solutions

$$P = \{x \in \mathbb{R}^3 \mid A \cdot x = b,\ x \geq 0\}\ .$$

$(\Leftrightarrow)\ x_1 + x_2 + x_3 = 1$

$x_1, x_2, x_3 \geq 0$



Feasible solutions lie on a 2-dimensional affine subspace of $\mathbb{R}^3$ and are only constraint by non-negativity constraints.

Def:

- Linear Subspace $S \subseteq \mathbb{R}^n$

  if $x, y \in S$ and $a, b \in \mathbb{R}$

  $\Rightarrow ax + by \in S$

- Affine Subspace S_A: translate (shift) by some vector z

$$S_A = \{ x + z \mid x \in S \}$$

Visualizing LPs in Standard Form

More general:

- if $A \in \mathbb{R}^{m \times n}$ with $m \leq n$ and the rows of A are linearly independent, then

$$\{x \in \mathbb{R}^n \mid A \cdot x = b\}$$

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- set of feasible solutions lies in this affine subspace and is only constrained by non-negativity constraints $x \geq 0$.