COMP331/557

Chapter 2:
The Geometry of Linear Programming

(Bertsimas & Tsitsiklis, Chapter 2)

Polyhedra and Polytopes

Definition 2.1.

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

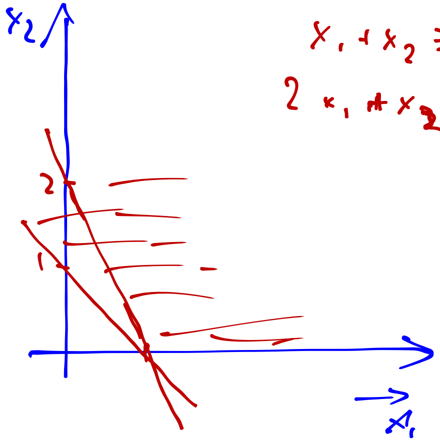
- a set $\{x \in \mathbb{R}^n \mid A \cdot x \geq b\}$ is called polyhedron

Polyhedra and Polytopes

Definition 2.1.

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

- a set $\{x \in \mathbb{R}^n \mid A \cdot x \geq b\}$ is called polyhedron
- b $\{x \mid A \cdot x = b,\ x \geq 0\}$ is polyhedron in standard form representation



$x_1 + x_2 \geq 1$

$2x_1 + x_2 \geq 2$

Polyhedra and Polytopes

Definition 2.1.

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

a. set $\{x \in \mathbb{R}^n \mid A \cdot x \geq b\}$ is called polyhedron
b. $\{x \mid A \cdot x = b,\ x \geq 0\}$ is polyhedron in standard form representation

Definition 2.2.

a. Set $S \subseteq \mathbb{R}^n$ is bounded if there is $K \in \mathbb{R}$ such that

$$\underbrace{\|x\|_\infty}_{= \max_i x_i} \leq K \qquad \text{for all } x \in S.$$

Def: $\|x\|_p = \sqrt[p]{\sum_i x_i^p}$

Polyhedra and Polytopes

Definition 2.1.

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

a. set $\{x \in \mathbb{R}^n \mid A \cdot x \geq b\}$ is called polyhedron
b. $\{x \mid A \cdot x = b,\ x \geq 0\}$ is polyhedron in standard form representation

Definition 2.2.

a. Set $S \subseteq \mathbb{R}^n$ is bounded if there is $K \in \mathbb{R}$ such that

$$\|x\|_\infty \leq K \qquad \text{for all } x \in S.$$

b. A bounded polyhedron is called polytope.

Hyperplanes and Halfspaces

Definition 2.3.

Let $a \in \mathbb{R}^n \setminus \{0\}$ and $b \in \mathbb{R}$:

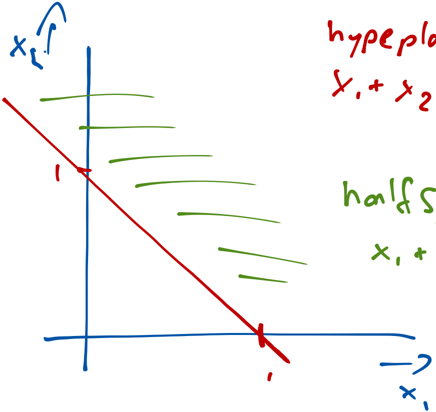
- a set $\{x \in \mathbb{R}^n \mid a^T \cdot x = b\}$ is called hyperplane

Hyperplanes and Halfspaces

Definition 2.3.

Let $a \in \mathbb{R}^n \setminus \{0\}$ and $b \in \mathbb{R}$:

a. set $\{x \in \mathbb{R}^n \mid a^T \cdot x = b\}$ is called hyperplane
b. set $\{x \in \mathbb{R}^n \mid a^T \cdot x \geq b\}$ is called halfspace



hyperplane
$x_1 + x_2 = 1$

halfspace
$x_1 + x_2 \geq 1$

Aside:

vector a is perpendicular to halfspace

Hyperplanes and Halfspaces

Definition 2.3.

Let $a \in \mathbb{R}^n \setminus \{0\}$ and $b \in \mathbb{R}$:

a set $\{x \in \mathbb{R}^n \mid a^T \cdot x = b\}$ is called hyperplane

b set $\{x \in \mathbb{R}^n \mid a^T \cdot x \geq b\}$ is called halfspace

Remarks

- Hyperplanes and halfspaces are convex sets.

Hyperplanes and Halfspaces

Definition 2.3.

Let $a \in \mathbb{R}^n \setminus \{0\}$ and $b \in \mathbb{R}$:

a. set $\{x \in \mathbb{R}^n \mid a^T \cdot x = b\}$ is called hyperplane
b. set $\{x \in \mathbb{R}^n \mid a^T \cdot x \geq b\}$ is called halfspace

Remarks

- Hyperplanes and halfspaces are convex sets.
- A polyhedron is an intersection of finitely many halfspaces.

Convex Combination and Convex Hull

Definition 2.4.

Let $x^1, \ldots, x^k \in \mathbb{R}^n$ and $\lambda_1, \ldots, \lambda_k \in \mathbb{R}_{\geq 0}$ with $\lambda_1 + \cdots + \lambda_k = 1$.

a. The vector $\sum_{i=1}^k \lambda_i \cdot x^i$ is a convex combination of $x^1, \ldots, x^k$.

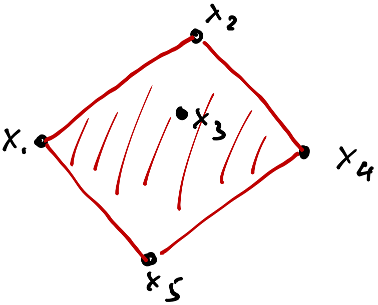
Convex Combination and Convex Hull

Definition 2.4.

Let $x^1, \ldots, x^k \in \mathbb{R}^n$ and $\lambda_1, \ldots, \lambda_k \in \mathbb{R}_{\geq 0}$ with $\lambda_1 + \cdots + \lambda_k = 1$.

a The vector $\sum_{i=1}^k \lambda_i \cdot x^i$ is a convex combination of $x^1, \ldots, x^k$.

b The convex hull of $x^1, \ldots, x^k$ is the set of all convex combinations.



nothing outside

Convex Sets, Convex Combinations, and Convex Hulls

Theorem 2.5.

a) The intersection of convex sets is convex.

Let X_i, $i \in I$ be convex sets.

Let $X := \bigcap_{i \in I} X_i$. Show that X is convex.

<u>To show</u>: $\forall x, y \in X \quad \forall \lambda \in [0,1]$

$$\lambda x + (1-\lambda) y \in X.$$

Let $x, y \in X \Rightarrow x, y \in X_i \quad \forall i$

Define $z = \lambda x + (1-\lambda) y$ for some $\lambda \in [0,1]$

$\Rightarrow z \in X_i \ \forall i$ (by convexity of X_i)

$\Rightarrow z \in X$ q.e.d. $\square$

Convex Sets, Convex Combinations, and Convex Hulls

Theorem 2.5.

a. The intersection of convex sets is convex.
b. Every polyhedron is a convex set.

Polyhedron = intersection of finitly many halfspaces

convex

Convex Sets, Convex Combinations, and Convex Hulls

Theorem 2.5.

a. The intersection of convex sets is convex.
b. Every polyhedron is a convex set.
c. A convex combination of a finite number of elements of a convex set also belongs to that set.

Convex Sets, Convex Combinations, and Convex Hulls

Theorem 2.5.

a. The intersection of convex sets is convex.
b. Every polyhedron is a convex set.
c. A convex combination of a finite number of elements of a convex set also belongs to that set.
d. The convex hull of finitely many vectors is a convex set.

Convex Sets, Convex Combinations, and Convex Hulls

Theorem 2.5.

a. The intersection of convex sets is convex.
b. Every polyhedron is a convex set.
c. A convex combination of a finite number of elements of a convex set also belongs to that set.
d. The convex hull of finitely many vectors is a convex set.

Corollary 2.6.

The convex hull of $x^1, \ldots, x^k \in \mathbb{R}^n$ is the smallest (w.r.t. inclusion) convex subset of $\mathbb{R}^n$ containing $x^1, \ldots, x^k$.

with respect to

Extreme Points and Vertices of Polyhedra

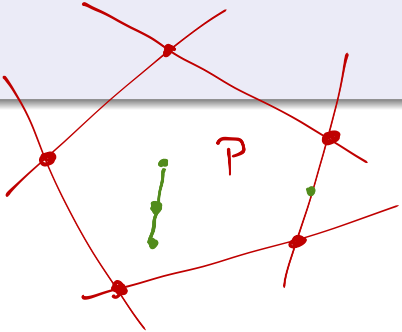
Definition 2.7.

Let $P \subseteq \mathbb{R}^n$ be a polyhedron.

a) $x \in P$ is an extreme point of P if

$$x \neq \lambda \cdot y + (1 - \lambda) \cdot z \qquad \text{for all } y, z \in P \setminus \{x\},\ 0 \leq \lambda \leq 1,$$

i. e., x is not a convex combination of two other points in P.



• extreme points

Extreme Points and Vertices of Polyhedra

Definition 2.7.

Let $P \subseteq \mathbb{R}^n$ be a polyhedron.

a $x \in P$ is an extreme point of P if

$$x \neq \lambda \cdot y + (1-\lambda) \cdot z \qquad \text{for all } y, z \in P \setminus \{x\},\ 0 \leq \lambda \leq 1,$$

i. e., x is not a convex combination of two other points in P.

b $x \in P$ is a vertex of P if there is some $c \in \mathbb{R}^n$ such that

$$c^T \cdot x < c^T \cdot y \qquad \text{for all } y \in P \setminus \{x\},$$

i. e., x is the unique optimal solution to the LP $\min\{c^T \cdot z \mid z \in P\}$.