Convex Sets, Convex Combinations, and Convex Hulls

Theorem 2.5.

a. The intersection of convex sets is convex.
b. Every polyhedron is a convex set.
c. A convex combination of a finite number of elements of a convex set also belongs to that set.
d. The convex hull of finitely many vectors is a convex set.

Corollary 2.6.

The convex hull of $x^1, \ldots, x^k \in \mathbb{R}^n$ is the smallest (w.r.t. inclusion) convex subset of $\mathbb{R}^n$ containing $x^1, \ldots, x^k$.

with respect to

Extreme Points and Vertices of Polyhedra

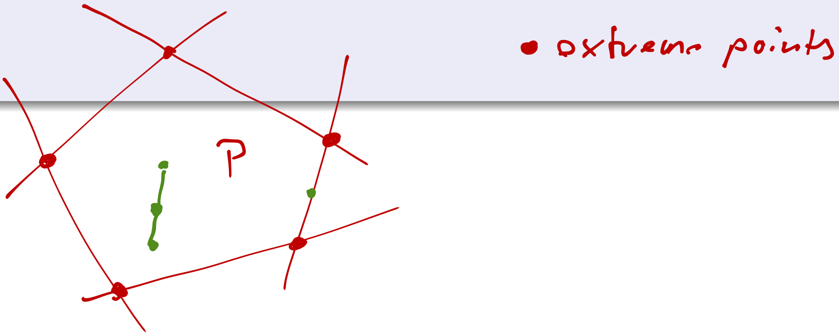
Definition 2.7.

Let $P \subseteq \mathbb{R}^n$ be a polyhedron.

a $x \in P$ is an extreme point of P if

$$x \neq \lambda \cdot y + (1-\lambda) \cdot z \qquad \text{for all } y, z \in P \setminus \{x\},\ 0 \leq \lambda \leq 1,$$

i. e., x is not a convex combination of two other points in P.



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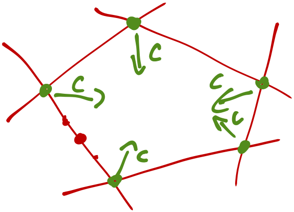
i. e., x is not a convex combination of two other points in P.

b) $x \in P$ is a vertex of P if there is some $c \in \mathbb{R}^n$ such that

$$c^T \cdot x < c^T \cdot y \qquad \text{for all } y \in P \setminus \{x\},$$

i. e., x is the unique optimal solution to the LP $\min\{c^T \cdot z \mid z \in P\}$.



Active and Binding Constraints

In the following, let $P \subseteq \mathbb{R}^n$ be a polyhedron defined by

$$\begin{aligned} {a_i}^T \cdot x &\geq b_i & \text{for } i \in M_1, \\ {a_i}^T \cdot x &= b_i & \text{for } i \in M_2, \end{aligned}$$

with $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$, for all i.

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with $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$, for all i.

Definition 2.8.

If $x^* \in \mathbb{R}^n$ satisfies ${a_i}^T \cdot x^* = b_i$ for some i, then the corresponding constraint is active (or binding) at x^*.

Basic Facts from Linear Algebra

Theorem 2.9.

Let $x^* \in \mathbb{R}^n$ and $I = \{i \mid {a_i}^T \cdot x^* = b_i\}$. The following are equivalent:

i. there are n vectors in $\{a_i \mid i \in I\}$ which are linearly independent;
ii. the vectors in $\{a_i \mid i \in I\}$ span $\mathbb{R}^n$;
iii. x^* is the unique solution to the system of equations ${a_i}^T \cdot x = b_i$, $i \in I$.

→ every element of $\mathbb{R}^n$ can be expressed as a linear combination of those vectors.

Aside: Show that $\begin{pmatrix}1\\0\\2\end{pmatrix}, \begin{pmatrix}2\\1\\4\end{pmatrix}, \begin{pmatrix}-1\\1\\2\end{pmatrix}$ are linear independent.

$$\begin{pmatrix}1 & 2 & -1\\0 & 1 & 1\\2 & 4 & 2\end{pmatrix} \xrightarrow{\text{Elementary row operations}} \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$

Vertices, Extreme Points, and Basic Feasible Solutions

Definition 2.10.

a $x^* \in \mathbb{R}^n$ is a basic solution of P if

- all equality constraints are active and
- there are n linearly independent constraints that are active.

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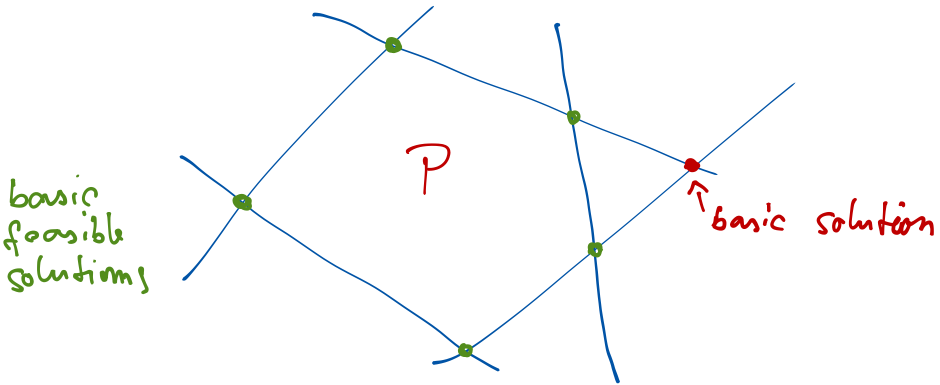

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Theorem 2.11.

For $x^* \in P$, the following are equivalent:

i. x^* is a vertex of P;
ii. x^* is an extreme point of P;
iii. x^* is a basic feasible solution of P.

Number of Vertices

Corollary 2.12.

a. A polyhedron has a finite number of vertices and basic solutions.

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- b For a polyhedron in $\mathbb{R}^n$ given by linear equations and m linear inequalities, this number is at most $\binom{m}{n}$.

$$\binom{m}{n} = \frac{m!}{n! \cdot (m-n)!}$$

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Example:

$P := \{x \in \mathbb{R}^n \mid 0 \leq x_i \leq 1, \ i = 1, \ldots, n\}$ (n-dimensional unit cube)

- number of constraints: $m = 2n$
- number of vertices: 2^n

$$\binom{2n}{n} = \frac{(2n)!}{n! \cdot n!} = \underbrace{\frac{2n}{n}}_{\geq 2} \cdot \underbrace{\frac{2n-1}{n-1}}_{\geq 2} \cdot \ldots \cdot \underbrace{\frac{n+1}{1}}_{\geq 2}$$

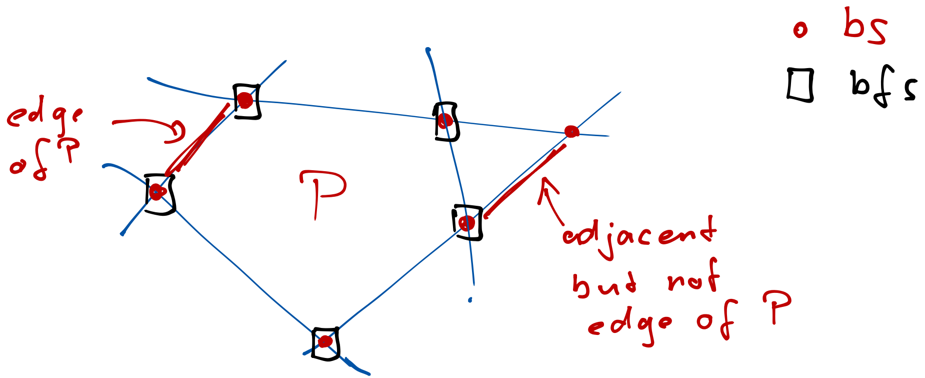
$$\geq 2^n$$

Adjacent Basic Solutions and Edges

Definition 2.13.

Let $P \subseteq \mathbb{R}^n$ be a polyhedron.

- **a** Two distinct basic solutions are adjacent if there are $n-1$ linearly independent constraints that are active at both of them.
- **b** If both solutions are feasible, the line segment that joins them is an edge of P.



Polyhedra in Standard Form

Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $P = \{x \in \mathbb{R}^n \mid A \cdot x = b,\ x \geq 0\}$.

Observation

One can assume without loss of generality that $\operatorname{rank}(A) = m$.

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Theorem 2.14.

$x \in \mathbb{R}^n$ is a basic solution of P if and only if $A \cdot x = b$ and there are indices $B(1), \ldots, B(m) \in \{1, \ldots, n\}$ such that

- columns $A_{B(1)}, \ldots, A_{B(m)}$ of matrix A are linearly independent and
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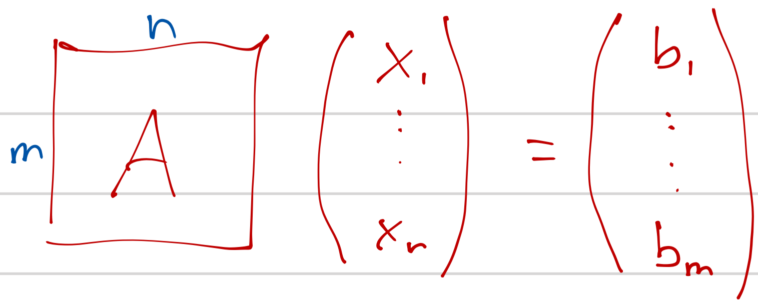
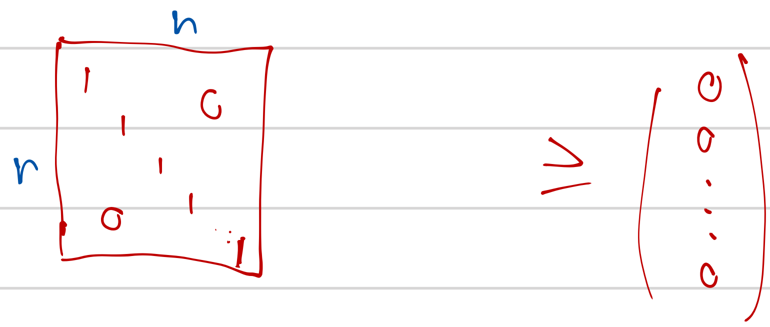
- $A_{B(1)}, \dots, A_{B(m)}$ are basic columns of A and form a basis of $\mathbb{R}^m$.
- The matrix $B := (A_{B(1)}, \dots, A_{B(m)}) \in \mathbb{R}^{m \times m}$ is called basis matrix.

m active constraint



$A \cdot x = b$

n−m active constraints



In basic solution x

n linear independent constraints

- m from $A \cdot x = b$
- n−m from $I_n \cdot x \geq 0$

These constraints are linear independent