Example:

Consider the following LP:

$$
\begin{array}{lrrrrrrrl}
\min & 2x_1 & & & +x_4 & & & +5x_7 & \\
\text{s.t.} & x_1 & +x_2 & +x_3 & +x_4 & & & & = 4 \\
 & x_1 & & & & +x_5 & & & = 2 \\
 & & & x_3 & & & +x_6 & & = 3 \\
 & & 3x_2 & +x_3 & & & & +x_7 & = 6 \\
 & & & & & & & x_j & \geq 0 \quad , \forall j
\end{array}
$$

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$$
(A|b) = \begin{pmatrix}
1 & 1 & 1 & 1 & & & & | & 4 \\
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\end{pmatrix}
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\end{pmatrix}
$$

A has full row rank $m = 4$.

Example:

Basis 1:

$$\left(\begin{array}{ccccccc|c} 1 & 1 & 1 & 1 & & & & 4 \\ 1 & & & & 1 & & & 2 \\ & & 1 & & & 1 & & 3 \\ & 3 & 1 & & & & 1 & 6 \end{array}\right)$$

$B(1)=4$, $B(2)=5$, $B(3)=6$, $B(4)=7$

Basis 2:

$$\left(\begin{array}{ccccccc|c} 1 & 1 & 1 & 1 & & & & 4 \\ 1 & & & & 1 & & & 2 \\ & & 1 & & & 1 & & 3 \\ & 3 & 1 & & & & 1 & 6 \end{array}\right)$$

$B(1)=2$; $B(2)=5$

$B(3)=6$; $B(4)=7$

Basis 3:

$$\left(\begin{array}{ccccccc|c} 1 & 1 & 1 & 1 & & & & 4 \\ 1 & & & & 1 & & & 2 \\ & & 1 & & & 1 & & 3 \\ & 3 & 1 & & & & 1 & 6 \end{array}\right)$$

$B(1)=2$; $B(2)=1$

$B(3)=3$; $B(4)=7$

$$\longrightarrow B = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 0 & 1 & 1 \end{pmatrix}$$

Example:

- Every basis B is invertible and can be transformed into the identity matrix by elementary row operations and column permutations. (Gaussian elemination)

No transformation needed

B = identity matrix

Basis 1:

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Basic solution:

$x_4 = 4$ $\quad$ $x_5 = 2$

$x_6 = 3$ $\quad$ $x_7 = 6$

$x_i = 0$ otherwise

Example:

- Every basis B is invertible and can be transformed into the identity matrix by elementary row operations and column permutations. (Gaussian elemination)
- If we transform the whole expended matrix with these operations, we obtain a solution of $Ax = b$ by setting the basic variables to the transformed right-hand-side. Such a solution is called basic solution for basis B.

Basis 2:

$$\left(\begin{array}{ccccccc|c} 1 & 1 & 1 & 1 & & & & 4 \\ 1 & & & & 1 & & & 2 \\ & & 1 & & & 1 & & 3 \\ & 3 & 1 & & & & 1 & 6 \end{array}\right) \quad IV - 3I$$

$$\left(\begin{array}{ccccccc|c} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 4 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 3 \\ -3 & 0 & -2 & -3 & 0 & 0 & 1 & -6 \end{array}\right)$$

Basic solution:

$x_2 = 4$

$x_5 = 2$

$x_6 = 3$

$x_7 = -6$ → not a feasible solution

$x_i = 0$ otherwise

Example:

Basis 3:

$$\left(\begin{array}{ccccccc|c} 1 & 1 & 1 & 1 & & & & 4 \\ 1 & & & & 1 & & & 2 \\ & & 1 & & & 1 & & 3 \\ & 3 & 1 & & & & 1 & 6 \end{array}\right) \quad I - II - III$$

$$\left(\begin{array}{ccccccc|c} 0 & 1 & 0 & 1 & -1 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 3 \\ 0 & 3 & 1 & 0 & 0 & 0 & 1 & 6 \end{array}\right) \quad IV - 3I - III$$

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Basis solution:

$x_2 = -1$ → not feasible

$x_1 = 2$

$x_3 = 3$

$x_4 = 6$

$x_i = 0$ otherwise

Example:

Basis 3:

$$\left(\begin{array}{ccccccc|c} 1 & 1 & 1 & 1 & & & & 4 \\ 1 & & & & 1 & & & 2 \\ & & 1 & & & 1 & & 3 \\ & 3 & 1 & & & & 1 & 6 \end{array}\right)$$

Example:

- If we permute the columns of A and x such that $A = (A_B, A_N)$ and $x = \binom{x_B}{x_N}$, then the elementary transformations correspond to multiplying the linear system

$$(A_B, A_N)\begin{pmatrix} x_B \\ x_N \end{pmatrix} = b$$

from the left with the inverse B^{-1} of the basis:

$$\begin{aligned} && B^{-1}(A_B, A_N)\begin{pmatrix} x_B \\ x_N \end{pmatrix} &= B^{-1}b \\ \Leftrightarrow && B^{-1}A_B x_B + B^{-1}A_N x_N &= B^{-1}b \\ \Leftrightarrow && x_B + B^{-1}A_N x_N &= B^{-1}b \end{aligned}$$

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- Setting $x_N = 0$, we obtain $x_B = B^{-1}b$.
- So if B is a basis, we obtain the associated basic solution $x = (x_B, x_N)^T$ as $x_B = B^{-1}b$, $x_N = 0$.

Example: Basis 3 — those were swapped.

$$B := \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 3 & 0 & 1 & 1 & 0 & 0 & & 1 \end{array}\right)$$