Example: Basis 3 — those were swapped.

$$B = \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 3 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \quad I - II - III$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 3 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \quad IV - 3I - III$$

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B^{-1}

$$x_B = \begin{pmatrix} x_2 \\ x_1 \\ x_3 \\ x_4 \end{pmatrix}$$

$$B^{-1} \cdot b = \begin{pmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \\ x_3 \\ x_4 \end{pmatrix}$$

Basic Columns and Basic Solutions

Observation 2.15.

Let $x \in \mathbb{R}^n$ be a basic solution, then:

- $B \cdot x_B = b$ and thus $x_B = B^{-1} \cdot b$;
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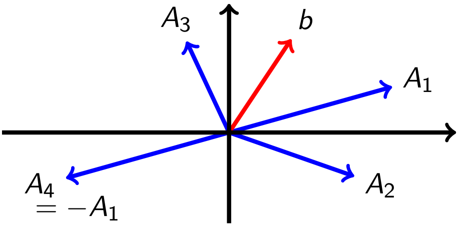
Example: $m = 2$



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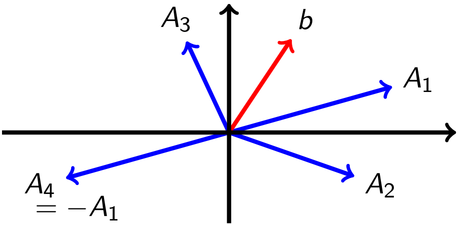

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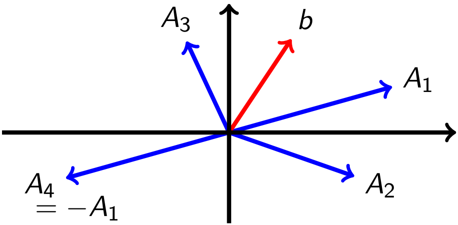

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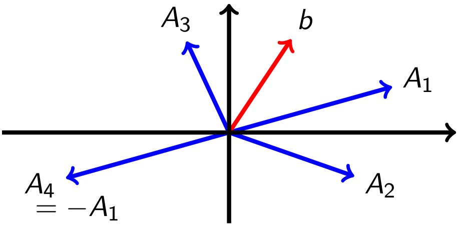

- A_1, A_3 or A_2, A_3 form bases with corresp. basic feasible solutions.
- A_1, A_4 do not form a basis.
- A_1, A_2 and A_2, A_4 and A_3, A_4 form bases with infeasible basic solution.

Bases and Basic Solutions

Corollary 2.16.

- Every basis $A_{B(1)}, \ldots, A_{B(m)}$ determines a unique basic solution.

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- Every basis $A_{B(1)}, \ldots, A_{B(m)}$ determines a unique basic solution.
- Thus, different basic solutions correspond to different bases.
- But: two different bases might yield the same basic solution.

Example: If $b = 0$, then $x = 0$ is the only basic solution.

Adjacent Bases

Definition 2.17.

Two bases $A_{B(1)}, \ldots, A_{B(m)}$ and $A_{B'(1)}, \ldots, A_{B'(m)}$ are adjacent if they share all but one column.

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Two bases $A_{B(1)}, \ldots, A_{B(m)}$ and $A_{B'(1)}, \ldots, A_{B'(m)}$ are adjacent if they share all but one column.

Observation 2.18.

a. Two adjacent basic solutions can always be obtained from two adjacent bases.
b. If two adjacent bases lead to distinct basic solutions, then the latter are adjacent.

Degeneracy

Definition 2.19.

A basic solution x of a polyhedron P is degenerate if more than n constraints are active at x.

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A basic solution x of a polyhedron P is degenerate if more than n constraints are active at x.

Observation 2.20.

Let $P = \{x \in \mathbb{R}^n \mid A \cdot x = b,\ x \geq 0\}$ be a polyhedron in standard form with $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

- a) A basic solution $x \in P$ is degenerate if and only if more than $n - m$ components of x are zero.
- b) For a non-degenerate basic solution $x \in P$, there is a unique basis.

Three Different Reasons for Degeneracy

- i redundant variables

Example:

$$\begin{aligned} x_1 + x_2 \quad &= 1 \\ x_3 &= 0 \\ x_1, x_2, x_3 &\geq 0 \end{aligned} \quad \longleftrightarrow \quad A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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$$\begin{aligned} x_1 + x_2 \quad &= 1 \\ {\color{red}x_3} \; &{\color{red}= 0} \\ x_1, x_2, x_3 &\geq 0 \end{aligned} \qquad \longleftrightarrow \qquad A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

ii redundant constraints

Example:

$$\begin{aligned} x_1 + 2\,x_2 &\leq 3 \\ 2\,x_1 + x_2 &\leq 3 \\ x_1 + x_2 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

$\Rightarrow 3x_1 + 3x_2 \leq 6$

redundant, implied by other constraints

Three Different Reasons for Degeneracy

1. redundant variables

   Example:

$$\begin{aligned} x_1 + x_2 \quad &= 1 \\ x_3 &= 0 \\ x_1,\, x_2,\, x_3 &\geq 0 \end{aligned} \quad \longleftrightarrow \quad A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2. redundant constraints

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iii. geometric reasons

Example: Octahedron

Three Different Reasons for Degeneracy

i. redundant variables

Example:

$$\begin{array}{rcrcl} x_1 & + & x_2 & & = 1 \\ & & & x_3 & = 0 \\ & & x_1,\ x_2,\ & x_3 & \geq 0 \end{array} \quad \longleftrightarrow \quad A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

ii. redundant constraints

Example:

$$\begin{array}{rcrl} x_1 & + & 2\,x_2 & \leq 3 \\ 2\,x_1 & + & x_2 & \leq 3 \\ x_1 & + & x_2 & \leq 2 \\ & & x_1,\ x_2 & \geq 0 \end{array}$$

iii. geometric reasons

Example: Octahedron

Observation 2.21.

Perturbing the right hand side vector b may remove degeneracy.