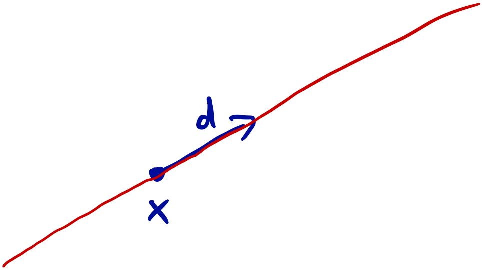
Existence of Extreme Points

Definition 2.22.

A polyhedron $P \subseteq \mathbb{R}^n$ contains a line if there is $x \in P$ and a direction $d \in \mathbb{R}^n \setminus \{0\}$ such that

$$x + \lambda \cdot d \in P \qquad \text{for all } \lambda \in \mathbb{R}.$$



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Thm 2.6 in book

Theorem 2.23.

Let $P = \{x \in \mathbb{R}^n \mid A \cdot x \geq b\} \neq \emptyset$ with $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. The following are equivalent:

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i. There exists an extreme point $x \in P$.
ii. P does not contain a line.

Existence of Extreme Points

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Theorem 2.23.

Let $P = \{x \in \mathbb{R}^n \mid A \cdot x \geq b\} \neq \emptyset$ with $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. The following are equivalent:

i There exists an extreme point $x \in P$.

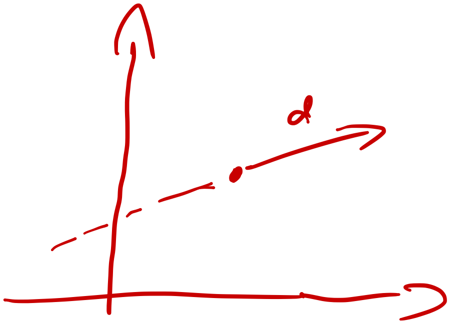
ii P does not contain a line.

iii A contains n linearly independent rows.

Existence of Extreme Points (cont.)

Corollary 2.24. = bounded polyhedron

- a) A non-empty polytope contains an extreme point.
- b) A non-empty polyhedron in standard form contains an extreme point.



Existence of Extreme Points (cont.)

Corollary 2.24.

a. A non-empty polytope contains an extreme point.

b. A non-empty polyhedron in standard form contains an extreme point.

Proof of b:

$$\begin{aligned} A \cdot x &= b \\ x &\geq 0 \end{aligned} \qquad \longleftrightarrow \qquad \begin{pmatrix} A \\ -A \\ I \end{pmatrix} \cdot x \geq \begin{pmatrix} b \\ -b \\ 0 \end{pmatrix}$$

□

Existence of Extreme Points (cont.)

Corollary 2.24.

- a) A non-empty polytope contains an extreme point.
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□

Example:

$$P = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{array}{rcrl} x_1 & + & x_2 & \geq 1 \\ x_1 & + & 2\,x_2 & \geq 0 \end{array} \right\}$$

Existence of Extreme Points (cont.)

Corollary 2.24.

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b. A non-empty polyhedron in standard form contains an extreme point.

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□

Example:

$$P = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{array}{rcrl} x_1 & + & x_2 & \geq 1 \\ x_1 & + & 2\,x_2 & \geq 0 \end{array} \right\}$$

contains a line since $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in P$ for all $\lambda \in \mathbb{R}$.

Optimality of Extreme Points

Theorem 2.25.

Let $P \subseteq \mathbb{R}^n$ a polyhedron and $c \in \mathbb{R}^n$. If P has an extreme point and $\min\{c^T \cdot x \mid x \in P\}$ is bounded, there is an extreme point that is optimal.

Optimality of Extreme Points

Theorem 2.25.

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Corollary 2.26.

Every linear programming problem is either infeasible or unbounded or there exists an optimal solution.

Optimality of Extreme Points

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Corollary 2.26.

Every linear programming problem is either infeasible or unbounded or there exists an optimal solution.

Proof: Every linear program is equivalent to an LP in standard form.
The claim thus follows from Corollary 2.24 and Theorem 2.25. □

Proof of Thm 2.25:

Assume P is non-empty.

Let Q be set of optimal solutions.

Let $P = \{ x \in \mathbb{R}^n \mid Ax \geq b \}$ and v be the value of the cost function $c^T x$ in optimum.

$\Rightarrow Q = \{ x \in \mathbb{R}^n \mid Ax \geq b, c^T x = v \}$.

Since $Q \subseteq P$ and P contains no line

$\Rightarrow$ Q contains no line

$\Rightarrow$ Q has an extreme point.

Let x^* be an extreme point of Q.

We will show that x^* is also an extreme point of P.

Suppose it is not.

$\Rightarrow \exists\, y, z \in P$, $y \neq x^*$, $z \neq x^*$ s.t.

$$\lambda y + (1-\lambda) z = x^* \text{ for some } \lambda \in [0,1]$$

It follows that

$$v = c^T x^* = \lambda c^T y + (1-\lambda) c^T z$$

By optimality of x^*:

$$c^T y \geq c^T x^* \quad \text{and} \quad c^T z \geq c^T x^*$$

$\Rightarrow c^T y = c^T z = c^T x^* = v$

$\Rightarrow z, y \in Q \longrightarrow$ contradicting to x^* being an extrem point of Q.

$\Rightarrow x^*$ is an extreme point of P. $\square$

COMP331/557

Chapter 3:
The Simplex Method

(Bertsimas & Tsitsiklis, Chapter 3)

Linear Program in Standard Form

Throughout this chapter, we consider the following standard form problem:

$$\begin{aligned} \text{minimize} \quad & c^T \cdot x \\ \text{subject to} \quad & A \cdot x = b \\ & x \geq 0 \end{aligned}$$

with $A \in \mathbb{R}^{m \times n}$, $\text{rank}(A) = m$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$.

Linear Program in Standard Form

Throughout this chapter, we consider the following standard form problem:

$$\begin{aligned} \text{minimize} \quad & c^T \cdot x \\ \text{subject to} \quad & A \cdot x = b \\ & x \geq 0 \end{aligned}$$

with $A \in \mathbb{R}^{m \times n}$, $\text{rank}(A) = m$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$.

Recall:

- Let $B = (A_{B(1)}, \ldots, A_{B(m)})$ be a basis matrix of A. Then B corresponds to the basic solution $x = (x_B, x_N)^T$, where $x_B = B^{-1}b$ and $x_N = 0$.
- $x = (x_B, x_N)^T$ is a basic feasible solution if $x_B \geq 0$.

Main Idea of the Simplex Method

Idea

Change basis by exchanging one basic column with one non-basic column.

More precisely:

- Start with a basis B defining a system with basic feasible solution.
- Then proceed in iterations. In each iteration:
  - select a nonbasic column j such that bringing j into the basis decreases (or at least does not increase) the value of the objective function. Stop, if no such column exists.
  - select a basic column ℓ such that exchanging columns j and ℓ maintain a basis with associated basic feasible solution
  - update the corresponding system

Iterations are called pivot steps.

Full Tableau Implementation: An Example

A simple linear programming problem:

$$\begin{array}{rrcrcrcl}
\min & -10\,x_1 & - & 12\,x_2 & - & 12\,x_3 & & \\
\text{s.t.} & x_1 & + & 2\,x_2 & + & 2\,x_3 & \leq & 20 \\
 & 2\,x_1 & + & x_2 & + & 2\,x_3 & \leq & 20 \\
 & 2\,x_1 & + & 2\,x_2 & + & x_3 & \leq & 20 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}$$

Set of Feasible Solutions

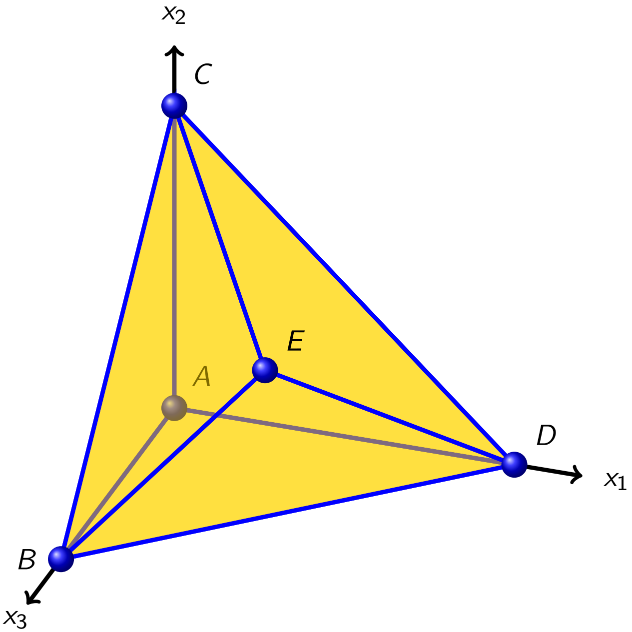
$A = (0, 0, 0)^T$

$B = (0, 0, 10)^T$

$C = (0, 10, 0)^T$

$D = (10, 0, 0)^T$

$E = (4, 4, 4)^T$



Introducing Slack Variables

$$
\begin{array}{lrcrcrcr}
\min & -10\,x_1 & - & 12\,x_2 & - & 12\,x_3 & & \\
\text{s.t.} & x_1 & + & 2\,x_2 & + & 2\,x_3 & \leq & 20 \\
 & 2\,x_1 & + & x_2 & + & 2\,x_3 & \leq & 20 \\
 & 2\,x_1 & + & 2\,x_2 & + & x_3 & \leq & 20 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Introducing Slack Variables

$$
\begin{array}{lrcrcrcr}
\min & -10\,x_1 & - & 12\,x_2 & - & 12\,x_3 & & \\
\text{s.t.} & x_1 & + & 2\,x_2 & + & 2\,x_3 & \leq & 20 \\
 & 2\,x_1 & + & x_2 & + & 2\,x_3 & \leq & 20 \\
 & 2\,x_1 & + & 2\,x_2 & + & x_3 & \leq & 20 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

LP in standard form

$$
\begin{array}{lrcrcrcrcrcrcr}
\min & -10\,x_1 & - & 12\,x_2 & - & 12\,x_3 & & & & & & & & \\
\text{s.t.} & x_1 & + & 2\,x_2 & + & 2\,x_3 & + & x_4 & & & & & = & 20 \\
 & 2\,x_1 & + & x_2 & + & 2\,x_3 & & & + & x_5 & & & = & 20 \\
 & 2\,x_1 & + & 2\,x_2 & + & x_3 & & & & & + & x_6 & = & 20 \\
 & & & & & & & & & & & x_1, \ldots, x_6 & \geq & 0
\end{array}
$$

Introducing Slack Variables

$$\begin{array}{lrcrcrcr}
\min & -10\,x_1 & - & 12\,x_2 & - & 12\,x_3 & & \\
\text{s.t.} & x_1 & + & 2\,x_2 & + & 2\,x_3 & \leq & 20 \\
 & 2\,x_1 & + & x_2 & + & 2\,x_3 & \leq & 20 \\
 & 2\,x_1 & + & 2\,x_2 & + & x_3 & \leq & 20 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}$$

LP in standard form

$$\begin{array}{lrcrcrcrcrcrcr}
\min & -10\,x_1 & - & 12\,x_2 & - & 12\,x_3 & & & & & & & & \\
\text{s.t.} & x_1 & + & 2\,x_2 & + & 2\,x_3 & + & x_4 & & & & & = & 20 \\
 & 2\,x_1 & + & x_2 & + & 2\,x_3 & & & + & x_5 & & & = & 20 \\
 & 2\,x_1 & + & 2\,x_2 & + & x_3 & & & & & + & x_6 & = & 20 \\
 & & & & & & & & & & & x_1, \ldots, x_6 & \geq & 0
\end{array}$$

Observation

The right hand side of the system is non-negative. Therefore the point $(0, 0, 0, 20, 20, 20)^T$ is a basic feasible solution and we can start the simplex method with basis $B(1) = 4$, $B(2) = 5$, $B(3) = 6$.

Setting Up the Simplex Tableau

$$
\begin{array}{rrrrrrrrcr}
\min & -10\,x_1 & -12\,x_2 & -12\,x_3 & & & & & & \\
\text{s.t.} & x_1 & +2\,x_2 & +2\,x_3 & +x_4 & & & & = & 20 \\
 & 2\,x_1 & +x_2 & +2\,x_3 & & +x_5 & & & = & 20 \\
 & 2\,x_1 & +2\,x_2 & +x_3 & & & +x_6 & & = & 20 \\
 & & & & & & x_1,\dots,x_6 & & \geq & 0
\end{array}
$$

with basic feasible solution: $\underbrace{x_1 = x_2 = x_3 = 0}_{\text{non-basic variables}}$, $\underbrace{x_4 = 20, x_5 = 20, x_6 = 20}_{\text{basic variables}}$.

Setting Up the Simplex Tableau

$$
\begin{array}{rrrrrrrrl}
\min & -10\,x_1 & -12\,x_2 & -12\,x_3 & & & & & \\
\text{s.t.} & x_1 & +2\,x_2 & +2\,x_3 & +x_4 & & & = & 20 \\
& 2\,x_1 & +x_2 & +2\,x_3 & & +x_5 & & = & 20 \\
& 2\,x_1 & +2\,x_2 & +x_3 & & & +x_6 & = & 20 \\
& & & & & & x_1,\dots,x_6 & \geq & 0
\end{array}
$$

with basic feasible solution: $\underbrace{x_1 = x_2 = x_3 = 0}_{\text{non-basic variables}},\ \underbrace{x_4 = 20, x_5 = 20, x_6 = 20}_{\text{basic variables}}$.

		x_1	x_2	x_3	x_4	x_5	x_6
	0	-10	-12	-12	0	0	0
$x_4 =$	20	1	2	2	1	0	0
$x_5 =$	20	2	1	2	0	1	0
$x_6 =$	20	2	2	1	0	0	1

Setting Up the Simplex Tableau

$$
\begin{array}{llllllll}
\min & -10\,x_1 & -12\,x_2 & -12\,x_3 & & & & \\
\text{s.t.} & x_1 & +2\,x_2 & +2\,x_3 & +x_4 & & & = 20 \\
 & 2\,x_1 & +x_2 & +2\,x_3 & & +x_5 & & = 20 \\
 & 2\,x_1 & +2\,x_2 & +x_3 & & & +x_6 & = 20 \\
 & & & & & & x_1, \dots, x_6 & \geq 0
\end{array}
$$

with basic feasible solution: $\underbrace{x_1 = x_2 = x_3 = 0}_{\text{non-basic variables}}$, $\underbrace{x_4 = 20, x_5 = 20, x_6 = 20}_{\text{basic variables}}$.

		x_1	x_2	x_3	x_4	x_5	x_6
	0	−10	−12	−12	0	0	0
$x_4 =$	20	1	2	2	1	0	0
$x_5 =$	20	2	1	2	0	1	0
$x_6 =$	20	2	2	1	0	0	1

Remark: Initialisation not always that easy. See next week.

Pivoting

		x_1	x_2	x_3	x_4	x_5	x_6
	0	-10	-12	-12	0	0	0
$x_4 =$	20	1	2	2	1	0	0
$x_5 =$	20	2	1	2	0	1	0
$x_6 =$	20	2	2	1	0	0	1

Pivoting

		x_1	x_2	x_3	x_4	x_5	x_6
	0	-10	-12	-12	0	0	0
$x_4 =$	20	1	2	2	1	0	0
$x_5 =$	20	2	1	2	0	1	0
$x_6 =$	20	2	2	1	0	0	1

- Determine pivot column
  - Which non-basic variable can we increase to improve objective value?

Pivoting

		x_1	x_2	x_3	x_4	x_5	x_6
	0	-10	-12	-12	0	0	0
$x_4 =$	20	1	2	2	1	0	0
$x_5 =$	20	2	1	2	0	1	0
$x_6 =$	20	2	2	1	0	0	1

- Determine pivot column
  - Which non-basic variable can we increase to improve objective value?
  - E. g., smallest subscript rule: $\bar{c}_1 < 0$ and x_1 enters the basis.

Pivoting

		x_1	x_2	x_3	x_4	x_5	x_6
	0	-10	-12	-12	0	0	0
$x_4 =$	20	1	2	2	1	0	0
$x_5 =$	20	2	1	2	0	1	0
$x_6 =$	20	2	2	1	0	0	1

$$x_1 \leq \frac{x_{B(i)}}{u_i}$$

- Determine pivot column
  - Which non-basic variable can we increase to improve objective value?
  - E. g., smallest subscript rule: $\bar{c}_1 < 0$ and x_1 enters the basis.
- Find pivot row. How large can we make x_1 and stay feasible?

Pivoting

	x_1	x_2	x_3	x_4	x_5	x_6	$x_1 \leq \frac{x_{B(i)}}{u_i}$
	0	-10	-12	-12	0	0	0	
$x_4 =$	20	1	2	2	1	0	0	$\Rightarrow x_1 \leq 20$
$x_5 =$	20	2	1	2	0	1	0	
$x_6 =$	20	2	2	1	0	0	1	

- Determine pivot column
  - Which non-basic variable can we increase to improve objective value?
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Pivoting

		x_1	x_2	x_3	x_4	x_5	x_6	$x_1 \leq \frac{x_{B(i)}}{u_i}$
	0	-10	-12	-12	0	0	0	
$x_4 =$	20	1	2	2	1	0	0	$\Rightarrow x_1 \leq 20$
$x_5 =$	20	2	1	2	0	1	0	$\Rightarrow x_1 \leq 10$
$x_6 =$	20	2	2	1	0	0	1	

- Determine pivot column
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  - E. g., smallest subscript rule: $\bar{c}_1 < 0$ and x_1 enters the basis.
- Find pivot row. How large can we make x_1 and stay feasible?

Pivoting

		x_1	x_2	x_3	x_4	x_5	x_6	$x_1 \leq \frac{x_{B(i)}}{u_i}$
	0	-10	-12	-12	0	0	0	
$x_4 =$	20	1	2	2	1	0	0	$\Rightarrow x_1 \leq 20$
$x_5 =$	20	2	1	2	0	1	0	$\Rightarrow x_1 \leq 10$
$x_6 =$	20	2	2	1	0	0	1	$\Rightarrow x_1 \leq 10$

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Pivoting

		x_1	x_2	x_3	x_4	x_5	x_6	$x_1 \leq \frac{x_{B(i)}}{u_i}$
	0	-10	-12	-12	0	0	0	
$x_4 =$	20	1	2	2	1	0	0	$\Rightarrow x_1 \leq 20$
$x_5 =$	20	2	1	2	0	1	0	$\Rightarrow x_1 \leq 10$
$x_6 =$	20	2	2	1	0	0	1	$\Rightarrow x_1 \leq 10$

- Determine pivot column
  - Which non-basic variable can we increase to improve objective value?
  - E. g., smallest subscript rule: $\bar{c}_1 < 0$ and x_1 enters the basis.
- Find pivot row. How large can we make x_1 and stay feasible?
  - Rows 2 and 3 both attain the minimum.

Pivoting

		x_1	x_2	x_3	x_4	x_5	x_6	$x_1 \le \frac{x_{B(i)}}{u_i}$
	0	-10	-12	-12	0	0	0	
$x_4 =$	20	1	2	2	1	0	0	$\Rightarrow x_1 \le 20$
$x_5 =$	20	2	1	2	0	1	0	$\Rightarrow x_1 \le 10$
$x_6 =$	20	2	2	1	0	0	1	$\Rightarrow x_1 \le 10$

- Determine pivot column
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  - E. g., smallest subscript rule: $\bar{c}_1 < 0$ and x_1 enters the basis.
- Find pivot row. How large can we make x_1 and stay feasible?
  - Rows 2 and 3 both attain the minimum.
  - Choose $i = 2$ with $B(i) = 5$. $\Longrightarrow$ x_5 leaves the basis.

Pivoting

		x_1	x_2	x_3	x_4	x_5	x_6
	0	-10	-12	-12	0	0	0
$x_4 =$	20	1	2	2	1	0	0
$x_5 =$	20	2	1	2	0	1	0
$x_6 =$	20	2	2	1	0	0	1

- Determine pivot column
  - Which non-basic variable can we increase to improve objective value?
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- Find pivot row. How large can we make x_1 and stay feasible?
  - Rows 2 and 3 both attain the minimum.
  - Choose $i = 2$ with $B(i) = 5$. $\implies$ x_5 leaves the basis.
- Perform basis change: Eliminate other entries in the pivot column.

Pivoting

		x_1	x_2	x_3	x_4	x_5	x_6
	100	0	-7	-2	0	5	0
$x_4 =$	20	1	2	2	1	0	0
$x_5 =$	20	2	1	2	0	1	0
$x_6 =$	20	2	2	1	0	0	1

- Determine pivot column
  - Which non-basic variable can we increase to improve objective value?
  - E. g., smallest subscript rule: $\bar{c}_1 < 0$ and x_1 enters the basis.
- Find pivot row. How large can we make x_1 and stay feasible?
  - Rows 2 and 3 both attain the minimum.
  - Choose $i = 2$ with $B(i) = 5$. $\implies$ x_5 leaves the basis.
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Pivoting

		x_1	x_2	x_3	x_4	x_5	x_6
	100	0	-7	-2	0	5	0
$x_4 =$	20	1	2	2	1	0	0
$x_5 =$	20	2	1	2	0	1	0
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  - Which non-basic variable can we increase to improve objective value?
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- Find pivot row. How large can we make x_1 and stay feasible?
  - Rows 2 and 3 both attain the minimum.
  - Choose $i = 2$ with $B(i) = 5$. $\implies$ x_5 leaves the basis.
- Perform basis change: Eliminate other entries in the pivot column.

Pivoting

		x_1	x_2	x_3	x_4	x_5	x_6
	100	0	-7	-2	0	5	0
$x_4 =$	10	0	1.5	1	1	-0.5	0
$x_5 =$	20	2	1	2	0	1	0
$x_6 =$	20	2	2	1	0	0	1

- Determine pivot column
  - Which non-basic variable can we increase to improve objective value?
  - E. g., smallest subscript rule: $\bar{c}_1 < 0$ and x_1 enters the basis.
- Find pivot row. How large can we make x_1 and stay feasible?
  - Rows 2 and 3 both attain the minimum.
  - Choose $i = 2$ with $B(i) = 5$. $\implies$ x_5 leaves the basis.
- Perform basis change: Eliminate other entries in the pivot column.

Pivoting

		x_1	x_2	x_3	x_4	x_5	x_6
	100	0	-7	-2	0	5	0
$x_4 =$	10	0	1.5	1	1	-0.5	0
$x_5 =$	20	2	1	2	0	1	0
$x_6 =$	20	2	2	1	0	0	1

- Determine pivot column
  - Which non-basic variable can we increase to improve objective value?
  - E. g., smallest subscript rule: $\bar{c}_1 < 0$ and x_1 enters the basis.
- Find pivot row. How large can we make x_1 and stay feasible?
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- Perform basis change: Eliminate other entries in the pivot column.
- Obtain new basic feasible solution $(10, 0, 0, 10, 0, 0)^T$ with cost -100.