## Next Iterations



- $\bar{c}_{2}, \bar{c}_{3}<0 \Longrightarrow$ two possible choices for pivot column.
- Choose $x_{3}$ to enter the new basis.
- $u_{3}<0 \Longrightarrow$ third row cannot be chosen as pivot row.
- Choose $x_{4}$ to leave basis.



## Geometric Interpretation in the Original Polyhedron

$$
\begin{aligned}
& A=(0,0,0)^{T} \\
& B=(0,0,10)^{T} \\
& C=(0,10,0)^{T} \\
& D=(10,0,0)^{T} \\
& E=(4,4,4)^{T}
\end{aligned}
$$



## Geometric Interpretation in the Original Polyhedron

$$
\begin{aligned}
& A=(0,0,0)^{T} \\
& B=(0,0,10)^{T} \\
& C=(0,10,0)^{T} \\
& D=(10,0,0)^{T} \\
& E=(4,4,4)^{T}
\end{aligned}
$$



## Geometric Interpretation in the Original Polyhedron

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\begin{aligned}
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& C=(0,10,0)^{T} \\
& D=(10,0,0)^{T} \\
& E=(4,4,4)^{T}
\end{aligned}
$$



## Geometric Interpretation in the Original Polyhedron

$$
\begin{aligned}
& A=(0,0,0)^{T} \\
& B=(0,0,10)^{T} \\
& C=(0,10,0)^{T} \\
& D=(10,0,0)^{T} \\
& E=(4,4,4)^{T}
\end{aligned}
$$



Next Iterations


Next Iterations

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ | $x_{5}$ | $x_{6}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 120 | 0 | -4 | 0 | 2 | 4 | 0 |
| $x_{3}=$10 |  |  |  |  |  |  |
| $x_{1}=$ | 0 | 1.5 | 1 | 1 | -0.5 | 0 |
| $x_{6}=$ | 1 | -1 | 0 | -1 | 1 | 0 |
| 10 | 0 | 2.5 | 0 | 1 | -1.5 | 1 |

Next Iterations

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ | $x_{5}$ | $x_{6}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $x_{3}=$120 0 -4 0 2 4 0 <br> $x_{1}=$ $\frac{x_{B(i)}}{u_{i}}$      <br> $x_{6}=$ 0 1.5 1 1 -0.5 0 <br>  1 -1 0 -1 1 0 <br> 10 0 2.5 0 1 -1.5 1 | 20 <br> 4 |  |  |  |  |  |

Next Iterations

|  |  |  | $x_{2}$ | $x_{3}$ | ${ }^{4}$ | $x_{5}$ | $x_{6}$ | $\frac{x_{B(i)}}{u_{i}}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 120 | 0 | -4 | 0 | 2 | 4 | 0 |  |  |
| $x_{3}=$ | 10 | 0 | 1.5 | 1 | 1 | -0.5 | 0 | $\frac{20}{3}$ |  |
| $x_{1}=$ | 0 | 1 | -1 | 0 | -1 | 1 | 0 | - |  |
| $x_{6}=$ | 10 | 0 | 2.5 | 0 | 1 | -1.5 | 1 | 4 | $<\frac{20}{3}$ |

## Next Iterations

|  |  | $x_{1}$ | $x_{2}$ | $\chi_{3}$ | $x_{4}$ | $x_{5}$ | $x_{6}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 120 | 0 | -4 | 0 | 2 | 4 | 0 |
| $x_{3}=$ | 10 | 0 | 1.5 | 1 | 1 | -0.5 | 0 |
| $x_{1}=$ | 0 | 1 | -1 | 0 | -1 | 1 | 0 |
| $x_{6}=$ | 10 | 0 | 2.5 | 0 | 1 | -1.5 | 1 |

$x_{2}$ enters the basis, $x_{6}$ leaves it.

## Next Iterations

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ | $x_{5}$ | $x_{6}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| $x_{3}=$120 0 -4 0 2 4 <br> 10 0 1.5 1 1 -0.5 <br> $x_{1}=$      <br> $x_{6}=$ 1 -1 0 -1 1 <br> 10 0 2.5 0 1 -1.5 <br> 1      |  |  |  |  |  |  |

$x_{2}$ enters the basis, $x_{6}$ leaves it. We get

| $\quad$ |
| :--- |
|  |
| $x_{1}$ |$=$| 136 | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ | $x_{5}$ | $x_{6}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $x_{3}=$ | 0 | 0 | 3.6 | 1.6 | 1.6 |  |
| $x_{1}=$ |  |  |  |  |  |  |
| $x_{2}=$ | 0 | 0 | $\subset$ | 0.4 | 0.4 | -0.6 |
| 4 | 1 | 0 | 0 | -0.6 | 0.4 | 0.4 |
| 4 | 0 | 1 | 0 | 0.4 | -0.6 | 0.4 |

## Next Iterations

$x_{2}$ enters the bysis, $x_{6}$ leaves it. We get

|  |  | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ | $\chi_{5}$ | $x_{6}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 136 | 0 | 0 | 0 | 3.6 | 1.6 | 1.6 |
| $x_{3}=$ | 4 | 0 | 0 | 1 | 0.4 | 0.4 | -0.6 |
| $x_{1}=$ | 4 | 1 | 0 | 0 | -0.6 | 0.4 | 0.4 |
| $x_{2}=$ | 4 | 0 | 1 | 0 | 0.4 | -0.6 | 0.4 |

and the reduced costs are all non-negative.

## Next Iterations

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ | $x_{5}$ | $x_{6}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| 120 | 0 | -4 | 0 | 2 | 4 | 0 |
| $x_{3}=$10 0 1.5 1 1 -0.5 <br> $x_{1}=$ 0 1 -1 0 -1 <br> 1 0     <br> $x_{6}=$ 0 2.5 0 1 -1.5 |  |  |  |  |  |  |

$x_{2}$ enters the basis, $x_{6}$ leaves it. We get

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ | $x_{5}$ | $x_{6}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $x_{3}=$ | 4 | 0 | 0 | 1 | 0.4 | 0.4 |
| $x_{1}=$ | 0 | 0 | 0 | 3.6 | 1.6 | 1.6 |
| $x_{2}=0$ | 1 | 0 | 0 | -0.6 | 0.4 | 0.4 |
| 4 | 0 | 1 | 0 | 0.4 | -0.6 | 0.4 |

and the reduced costs are all non-negative.
Thus ( $4,4,4,0,0,0$ ) is an optimal solution with cos -136.) corresponding to point $E=(4,4,4)$ in the original polyhedron.

## Next Iterations

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ | $x_{5}$ | $x_{6}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| 120 | 0 | -4 | 0 | 2 | 4 | 0 |
|  | $x_{3}=$ |  |  |  |  |  |
| $x_{1}=$ | 0 | 1.5 | 1 | 1 | -0.5 | 0 |
| $x_{6}=$ | 1 | -1 | 0 | -1 | 1 | 0 |
| 10 | 0 | 2.5 | 0 | 1 | -1.5 | 1 |

$x_{2}$ enters the basis, $x_{6}$ leaves it. We get

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ | $x_{5}$ | $x_{6}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| 136 | 0 | 0 | 0 | 3.6 | 1.6 | 1.6 |
| $x_{3}=$ |  |  |  |  |  |  |
| $x_{1}=$ |  |  |  |  |  |  |
| $x_{2}=$ | 0 | 0 | 1 | 0.4 | 0.4 | -0.6 |
| 4 | 1 | 0 | 0 | -0.6 | 0.4 | 0.4 |
| 4 | 0 | 1 | 0 | 0.4 | -0.6 | 0.4 |

and the reduced costs are all non-negative.
Thus ( $4,4,4,0,0,0$ ) is an optimal solution with cost -136 , corresponding to point $E=(4,4,4)$ in the original polyhedron. Why is this optimal?

All Iterations from Geometric Point of View


All Iterations from Geometric Point of View

$$
\begin{aligned}
& A=(0,0,0)^{T} \\
& B=(0,0,10)^{T} \\
& C=(0,10,0)^{T} \\
& D=(10,0,0)^{T} \\
& E=(4,4,4)^{T}
\end{aligned}
$$



All Iterations from Geometric Point of View

$$
\begin{aligned}
& A=(0,0,0)^{T} \\
& B=(0,0,10)^{T} \\
& C=(0,10,0)^{T} \\
& D=(10,0,0)^{T} \\
& E=(4,4,4)^{T}
\end{aligned}
$$



All Iterations from Geometric Point of View

$$
\begin{aligned}
& A=(0,0,0)^{T} \\
& B=(0,0,10)^{T} \\
& C=(0,10,0)^{T} \\
& D=(10,0,0)^{T} \\
& E=(4,4,4)^{T}
\end{aligned}
$$



## Cycling

Problem: If an LP is degenerate, the simplex method might end up in an infinite loop (cycling).

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Problem: If an LP is degenerate, the simplex method might end up in an infinite loop (cycling).

Example:

|  |  | $x_{1}$ | $x_{2}$ | $x_{3}$ |  | X5 $\times_{6} \times$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 3 | -3/4 | 20 | $-1 / 2$ | 6 | 0 |
| $x_{5}=$ | 0 | 1/4 | -8 | -1 | 9 | (1) 00 |
| $x_{6}=$ | 0 | 1/2 | -12 | $-1 / 2$ | 3 | 0 (1) 0 |
| $x_{7}=$ | 1 | 0 | 0 | 1 | 0 | 0 0 (1) |

## Cycling

Problem: If an LP is degenerate, the simplex method might end up in an infinite loop (cycling).

Example:

|  |  | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ | $x_{5}$ | $x_{6}$ | $x_{7}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 3 | -3/4 | 20 | -1/2 | 6 | 0 | 0 | 0 |
| $x_{5}=$ | 0 | 1/4 | -8 | -1 | 9 | 1 | 0 | 0 |
| $x_{6}=$ | 0 | 1/2 | -12 | $-1 / 2$ | 3 | 0 | 1 | 0 |
| $x_{7}=$ | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |

Pivoting rules

- Column selection: let nonbasic variable with most negative reduced cost $\bar{c}_{j}$ enter the basis, i. e., steepest descent rule.
- Row selection: among basic variables that are eligible to exit the basis, select the one with smallest subscript.


## Iteration 1

|  |  | $x_{1}$ | $x_{2}$ | $\times_{3}$ | $x_{4}$ | $x_{5}$ | $x_{6}$ | $x_{7}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 3 | -3/4 | 20 | $-1 / 2$ | 6 | 0 | 0 | 0 |
| $x_{5}=$ | 0 | 1/4 | -8 | -1 | 9 | 1 | 0 | 0 |
| $x_{6}=$ | 0 | 1/2 | -12 | $-1 / 2$ | 3 | 0 | 1 | 0 |
| $x_{7}=$ | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |

## Bases visited

$(5,6,7)$

## Iteration 1

|  |  | $x_{1}$ | $x_{2}$ | $\times_{3}$ | $x_{4}$ | $x_{5}$ | $x_{6}$ | $x_{7}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 3 | -3/4 | 20 | $-1 / 2$ | 6 | 0 | 0 | 0 |
| $x_{5}=$ | 0 | 1/4 | -8 | -1 | 9 | 1 | 0 | 0 |
| $x_{6}=$ | 0 | 1/2 | -12 | $-1 / 2$ | 3 | 0 | 1 | 0 |
| $x_{7}=$ | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |

## Bases visited

$(5,6,7)$

## Iteration 1

|  |  | $x_{1}$ | $x_{2}$ | $x_{3}$ | $\chi_{4}$ | $\times_{5}$ | $x_{6}$ | $x_{7}$ | $\frac{x_{B(i)}}{u_{i}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 3 | -3/4 | 20 | $-1 / 2$ | 6 | 0 | 0 | 0 |  |
| $x_{5}=$ | 0 | 1/4 | -8 | -1 | 9 | 1 | 0 | 0 | 0 |
| $x_{6}=$ | 0 | 1/2 | -12 | $-1 / 2$ | 3 | 0 | 1 | 0 | 0 |
| $x_{7}=$ | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | - |

## Bases visited

$(5,6,7)$

## Iteration 1

|  |  | $x_{1}$ | $x_{2}$ | $x_{3}$ | $\chi_{4}$ | $x_{5}$ | $x_{6}$ | $x_{7}$ | $\frac{x_{B(i)}}{u_{i}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 33 | $-3 / 40$ | 20-4 | -1/2-3.5 | 633 | 03 | 00 | 00 |  |
| $x_{5}=$ | 0 | 1/4 | -8 | -1 | 9 | 1 | 0 | 0 | 0 |
| $x_{6}=$ | 0 | 1/2 | -12 | $-1 / 2$ | 3 | 0 | 1 | 0 | 0 |
| $x_{7}=$ | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | - |

Basis change: $x_{1}$ enters the basis $x_{5}$ leaves.

## Bases visited

$(5,6,7)$

## Iteration 1

|  |
| :--- |
| $x_{5}=$ $x_{1}$ $x_{2}$ $x_{3}$ $x_{4}$ $x_{5}$ $x_{6}$ $x_{7}$ <br> $x_{5}$ $-3 / 4$ 20 $-1 / 2$ 6 0 0 0 <br> $x_{6}=$ $1 / 4$ -8 -1 9 1 0 0 <br> $x_{7}=$ $1 / 2$ -12 $-1 / 2$ 3 0 1 0 <br> 0 1 0 0 1 0 0 0 <br> 1        |

Basis change: $x_{1}$ enters the basis $x_{5}$ leaves.

## Bases visited <br> $(5,6,7)$

## Iteration 1

|  |  | $x_{1}$ | $x_{2}$ | $x_{3}$ | ${ }^{4}$ | $x_{5}$ | $x_{6}$ | $x_{7}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 3 | 0 | -4 | -7/2 | 33 | 3 | 0 | 0 |
| $x_{5}=$ | 0 | 1/4 | -8 | -1 | 9 | 1 | 0 | 0 |
| $x_{6}=$ | 0 | (12) | -12 | $-1 / 2$ | 3 | 0 | 1 | 0 |
| $x_{7}=$ | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |

Basis change: $x_{1}$ enters the basis $x_{5}$ leaves.

## Bases visited <br> $(5,6,7)$

## Iteration 1

|  |  | $x_{1}$ | $x_{2}$ | $x_{3}$ | ${ }^{4}$ | $x_{5}$ | $x_{6}$ | $x_{7}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 3 | 0 | -4 | $-7 / 2$ | 33 | 3 | 0 | 0 |
| $x_{5}=$ | 0 | 1/4 | -8 | -1 | 9 | 1 | 0 | 0 |
| $x_{6}=$ | 0 | 1/2 | -12 | -1/2 | 3 | 0 | 1 | 0 |
| $x_{7}=$ | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |

Basis change: $x_{1}$ enters the basis $x_{5}$ leaves.

## Bases visited <br> $(5,6,7)$

## Iteration 1

|  |  | $x_{1}$ | $x_{2}$ | $x_{3}$ | ${ }^{4}$ | $\chi_{5}$ | $x_{6}$ | $x_{7}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 3 | 0 | -4 | $-7 / 2$ | 33 | 3 | 0 | 0 |
| $x_{5}=$ | 0 | 1/4 | -8 | -1 | 9 | 1 | 0 | 0 |
| $x_{6}=$ | 0 | 0 | 4 | 3/2 | -15 | -2 | 1 | 0 |
| $x_{7}=$ | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |

Basis change: $x_{1}$ enters the basis $x_{5}$ leaves.

## Bases visited <br> $(5,6,7)$

## Iteration 1

|  |  | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ | $x_{5}$ | $x_{6}$ | $x_{7}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 3 | 0 | -4 | -7/2 | 33 | 3 | 0 | 0 |
| $x_{5}=$ | 0 | 1/4 | -8 | -1 | 9 | 1 | 0 | 0 |
| $x_{6}=$ | 0 | 0 | 4 | 3/2 | -15 | -2 | 1 | 0 |
| $x_{7}=$ | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |

Basis change: $x_{1}$ enters the basis $x_{5}$ leaves.

## Bases visited <br> $(5,6,7)$

## Iteration 1

|  |  | $x_{1}$ | $x_{2}$ | $x_{3}$ | ${ }^{4}$ | $x_{5}$ | $x_{6}$ | $x_{7}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 3 | 0 | -4 | $-7 / 2$ | 33 | 3 | 0 | 0 |
| $x_{5}=$ | 0 | 1/4 | -8 | -1 | 9 | 1 | 0 | 0 |
| $x_{6}=$ | 0 | 0 | 4 | 3/2 | -15 | -2 | 1 | 0 |
| $x_{7}=$ | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |

Basis change: $x_{1}$ enters the basis $x_{5}$ leaves.

## Bases visited <br> $(5,6,7)$

## Iteration 1

|  |  | $x_{1}$ | $x_{2}$ | $\times_{3}$ | $x_{4}$ | $x_{5}$ | $x_{6}$ | $x_{7}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 3 | 0 | -4 | $-7 / 2$ | 33 | 3 | 0 | 0 |
| $x_{1}=$ | 0 | (1) | -32 | -4 | 36 | 4 | 0 | 0 |
| $x_{6}=$ | 0 | 0 | 4 | 3/2 | -15 | -2 | (1) | 0 |
| $x_{7}=$ | 1 | 0 | 0 | 1 | 0 | 0 | 0 | (1) |

Basis change: $x_{1}$ enters the basis $x_{5}$ leaves.

## Bases visited <br> $(5,6,7)$

## Iteration 2

|  |  | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ | $x_{5}$ | $x_{6}$ | $x_{7}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 3 | 0 | -4 | -7/2 | 33 | 3 | 0 | 0 |
| $x_{1}=$ | 0 | 1 | -32 | -4 | 36 | 4 | 0 | 0 |
| $x_{6}=$ | 0 | 0 | 4 | 3/2 | -15 | -2 | 1 | 0 |
| $x_{7}=$ | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |

## Bases visited

$(5,6,7) \rightarrow(1,6,7)$

