Iteration 2

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
	3	0	-4	$-7/2$	33	3	0	0
$x_1 =$	0	1	-32	-4	36	4	0	0
$x_6 =$	0	0	4	$3/2$	-15	-2	1	0
$x_7 =$	1	0	0	1	0	0	0	1

Bases visited

$(5, 6, 7) \rightarrow (1, 6, 7)$

Iteration 2

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
	3	0	-4	$-7/2$	33	3	0	0
$x_1 =$	0	1	-32	-4	36	4	0	0
$x_6 =$	0	0	4	$3/2$	-15	-2	1	0
$x_7 =$	1	0	0	1	0	0	0	1

Bases visited

$(5, 6, 7) \rightarrow (1, 6, 7)$

Iteration 2

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\frac{x_{B(i)}}{u_i}$
	3	0	-4	$-7/2$	33	3	0	0	
$x_1 =$	0	1	-32	-4	36	4	0	0	–
$x_6 =$	0	0	4	$3/2$	-15	-2	1	0	0
$x_7 =$	1	0	0	1	0	0	0	1	–

Bases visited

$(5, 6, 7) \rightarrow (1, 6, 7)$

Iteration 2

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\frac{x_{B(i)}}{u_i}$
	3	0	-4	$-7/2$	33	3	0	0	
$x_1 =$	0	1	-32	-4	36	4	0	0	–
$x_6 =$	0	0	**4**	$3/2$	-15	-2	1	0	0
$x_7 =$	1	0	0	1	0	0	0	1	–

Basis change: x_2 enters the basis x_6 leaves.

Bases visited

$(5, 6, 7) \rightarrow (1, 6, 7)$

Iteration 3

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
	3	0	0	-2	18	1	1	0
$x_1 =$	0	1	0	8	-84	-12	8	0
$x_2 =$	0	0	1	$3/8$	$-15/4$	$-1/2$	$1/4$	0
$x_7 =$	1	0	0	1	0	0	0	1

$x_3 \leq \frac{0}{8} = 0$

$x_3 \leq 0$

$x_3 \leq 1$

Bases visited

$(5, 6, 7) \rightarrow (1, 6, 7) \rightarrow (1, 2, 7)$

Iteration 3

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
	3	0	0	-2	18	1	1	0
$x_1 =$	0	1	0	8	-84	-12	8	0
$x_2 =$	0	0	1	$3/8$	$-15/4$	$-1/2$	$1/4$	0
$x_7 =$	1	0	0	1	0	0	0	1

Bases visited

$(5, 6, 7) \rightarrow (1, 6, 7) \rightarrow (1, 2, 7)$

Iteration 3

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\frac{x_{B(i)}}{u_i}$
	3	0	0	-2	18	1	1	0	
$x_1 =$	0	1	0	8	-84	-12	8	0	0
$x_2 =$	0	0	1	$3/8$	$-15/4$	$-1/2$	$1/4$	0	0
$x_7 =$	1	0	0	1	0	0	0	1	1

Bases visited

$(5,6,7) \rightarrow (1,6,7) \rightarrow (1,2,7)$

Iteration 3

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\frac{x_{B(i)}}{u_i}$
	3	0	0	-2	18	1	1	0	
$x_1 =$	0	1	0	**8**	-84	-12	8	0	0
$x_2 =$	0	0	1	$3/8$	$-15/4$	$-1/2$	$1/4$	0	0
$x_7 =$	1	0	0	1	0	0	0	1	1

Basis change: x_3 enters the basis x_1 leaves.

Bases visited

$(5,6,7) \rightarrow (1,6,7) \rightarrow (1,2,7)$

Iteration 4

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
	3	$1/4$	0	0	-3	-2	3	0
$x_3 =$	0	$1/8$	0	1	$-21/2$	$-3/2$	1	0
$x_2 =$	0	$-3/64$	1	0	$3/16$	$1/16$	$-1/8$	0
$x_7 =$	1	$-1/8$	0	0	$21/2$	$3/2$	-1	1

Bases visited

$(5,6,7) \rightarrow (1,6,7) \rightarrow (1,2,7) \rightarrow (3,2,7)$

Iteration 4

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
	3	$1/4$	0	0	-3	-2	3	0
$x_3 =$	0	$1/8$	0	1	$-21/2$	$-3/2$	1	0
$x_2 =$	0	$-3/64$	1	0	$3/16$	$1/16$	$-1/8$	0
$x_7 =$	1	$-1/8$	0	0	$21/2$	$3/2$	-1	1

Bases visited

$(5,6,7) \rightarrow (1,6,7) \rightarrow (1,2,7) \rightarrow (3,2,7)$

Iteration 4

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\frac{x_{B(i)}}{u_i}$
	3	1/4	0	0	−3	−2	3	0	
$x_3 =$	0	1/8	0	1	−21/2	−3/2	1	0	−
$x_2 =$	0	−3/64	1	0	3/16	1/16	−1/8	0	0
$x_7 =$	1	−1/8	0	0	21/2	3/2	−1	1	2/21

Bases visited

$(5, 6, 7) \rightarrow (1, 6, 7) \rightarrow (1, 2, 7) \rightarrow (3, 2, 7)$

Iteration 4

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\frac{x_{B(i)}}{u_i}$
	3	1/4	0	0	−3	−2	3	0	
$x_3 =$	0	1/8	0	1	−21/2	−3/2	1	0	–
$x_2 =$	0	−3/64	1	0	**3/16**	1/16	−1/8	0	0
$x_7 =$	1	−1/8	0	0	21/2	3/2	−1	1	2/21

Basis change: x_4 enters the basis x_2 leaves.

Bases visited

$(5, 6, 7) \rightarrow (1, 6, 7) \rightarrow (1, 2, 7) \rightarrow (3, 2, 7)$

Iteration 5

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
	3	$-1/2$	16	0	0	-1	1	0
$x_3 =$	0	$-5/2$	56	1	0	2	-6	0
$x_4 =$	0	$-1/4$	$16/3$	0	1	$1/3$	$-2/3$	0
$x_7 =$	1	$5/2$	-56	0	0	-2	6	1

Bases visited

$(5,6,7) \rightarrow (1,6,7) \rightarrow (1,2,7) \rightarrow (3,2,7) \rightarrow (3,4,7)$

Iteration 5

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
	3	$-1/2$	16	0	0	-1	1	0
$x_3 =$	0	$-5/2$	56	1	0	2	-6	0
$x_4 =$	0	$-1/4$	$16/3$	0	1	$1/3$	$-2/3$	0
$x_7 =$	1	$5/2$	-56	0	0	-2	6	1

Bases visited

$(5,6,7) \rightarrow (1,6,7) \rightarrow (1,2,7) \rightarrow (3,2,7) \rightarrow (3,4,7)$

Observation

After 4 pivoting iterations our basic feasible solution still has not changed.

Iteration 5

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
	3	$-1/2$	16	0	0	-1	1	0
$x_3 =$	0	$-5/2$	56	1	0	2	-6	0
$x_4 =$	0	$-1/4$	$16/3$	0	1	$1/3$	$-2/3$	0
$x_7 =$	1	$5/2$	-56	0	0	-2	6	1

Bases visited

$(5,6,7) \rightarrow (1,6,7) \rightarrow (1,2,7) \rightarrow (3,2,7) \rightarrow (3,4,7)$

Observation

After 4 pivoting iterations our basic feasible solution still has not changed.

Iteration 5

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\frac{x_{B(i)}}{u_i}$
	3	$-1/2$	16	0	0	-1	1	0	
$x_3 =$	0	$-5/2$	56	1	0	2	-6	0	0
$x_4 =$	0	$-1/4$	$16/3$	0	1	$1/3$	$-2/3$	0	0
$x_7 =$	1	$5/2$	-56	0	0	-2	6	1	$-$

Basis change: x_5 enters the basis x_3 leaves.

Bases visited

$(5,6,7) \rightarrow (1,6,7) \rightarrow (1,2,7) \rightarrow (3,2,7) \rightarrow (3,4,7)$

Observation

After 4 pivoting iterations our basic feasible solution still has not changed.

Iteration 5

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\frac{x_{B(i)}}{u_i}$
	3	$-1/2$	16	0	0	-1	1	0	
$x_3 =$	0	$-5/2$	56	1	0	**2**	-6	0	0
$x_4 =$	0	$-1/4$	$16/3$	0	1	$1/3$	$-2/3$	0	0
$x_7 =$	1	$5/2$	-56	0	0	-2	6	1	$-$

Basis change: x_5 enters the basis x_3 leaves.

Bases visited

$(5,6,7) \rightarrow (1,6,7) \rightarrow (1,2,7) \rightarrow (3,2,7) \rightarrow (3,4,7)$

Observation

After 4 pivoting iterations our basic feasible solution still has not changed.

Iteration 6

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
	3	$-7/4$	44	$1/2$	0	0	-2	0
$x_5 =$	0	$-5/4$	28	$1/2$	0	1	-3	0
$x_4 =$	0	$1/6$	-4	$-1/6$	1	0	$1/3$	0
$x_7 =$	1	0	0	1	0	0	0	1

Bases visited

$(5,6,7) \rightarrow (1,6,7) \rightarrow (1,2,7) \rightarrow (3,2,7) \rightarrow (3,4,7)$
$\rightarrow (5,4,7)$

Iteration 6

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
	3	$-7/4$	44	$1/2$	0	0	-2	0
$x_5 =$	0	$-5/4$	28	$1/2$	0	1	-3	0
$x_4 =$	0	$1/6$	-4	$-1/6$	1	0	$1/3$	0
$x_7 =$	1	0	0	1	0	0	0	1

Bases visited

$(5,6,7) \rightarrow (1,6,7) \rightarrow (1,2,7) \rightarrow (3,2,7) \rightarrow (3,4,7)$
$\rightarrow (5,4,7)$

Iteration 6

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\frac{x_{B(i)}}{u_i}$
	3	$-7/4$	44	$1/2$	0	0	-2	0	
$x_5 =$	0	$-5/4$	28	$1/2$	0	1	-3	0	$-$
$x_4 =$	0	$1/6$	-4	$-1/6$	1	0	$1/3$	0	0
$x_7 =$	1	0	0	1	0	0	0	1	$-$

Bases visited

$(5,6,7) \rightarrow (1,6,7) \rightarrow (1,2,7) \rightarrow (3,2,7) \rightarrow (3,4,7)$
$\rightarrow (5,4,7)$

Iteration 6

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\frac{x_{B(i)}}{u_i}$
	3	$-7/4$	44	$1/2$	0	0	-2	0	
$x_5 =$	0	$-5/4$	28	$1/2$	0	1	-3	0	–
$x_4 =$	0	$1/6$	-4	$-1/6$	1	0	$1/3$	0	0
$x_7 =$	1	0	0	1	0	0	0	1	–

Basis change: x_6 enters the basis x_4 leaves.

Bases visited

$(5,6,7) \rightarrow (1,6,7) \rightarrow (1,2,7) \rightarrow (3,2,7) \rightarrow (3,4,7)$
$\rightarrow (5,4,7)$

Back at the Beginning

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
	3	$-3/4$	20	$-1/2$	6	0	0	0
$x_5 =$	0	$1/4$	-8	-1	9	1	0	0
$x_6 =$	0	$1/2$	-12	$-1/2$	3	0	1	0
$x_7 =$	1	0	0	1	0	0	0	1

Bases visited

$(5,6,7) \rightarrow (1,6,7) \rightarrow (1,2,7) \rightarrow (3,2,7) \rightarrow (3,4,7)$
$\rightarrow (5,4,7) \rightarrow (5,6,7)$

Back at the Beginning

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
	3	$-3/4$	20	$-1/2$	6	0	0	0
$x_5 =$	0	$1/4$	-8	-1	9	1	0	0
$x_6 =$	0	$1/2$	-12	$-1/2$	3	0	1	0
$x_7 =$	1	0	0	1	0	0	0	1

Bases visited

$(5,6,7) \to (1,6,7) \to (1,2,7) \to (3,2,7) \to (3,4,7)$
$\to (5,4,7) \to (5,6,7)$

This is the same basis that we started with.

Back at the Beginning

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
	3	$-3/4$	20	$-1/2$	6	0	0	0
$x_5 =$	0	$1/4$	-8	-1	9	1	0	0
$x_6 =$	0	$1/2$	-12	$-1/2$	3	0	1	0
$x_7 =$	1	0	0	1	0	0	0	1

Bases visited

$(5,6,7) \rightarrow (1,6,7) \rightarrow (1,2,7) \rightarrow (3,2,7) \rightarrow (3,4,7)$
$\rightarrow (5,4,7) \rightarrow (5,6,7)$

This is the same basis that we started with.

Conclusion

Continuing with the pivoting rules we agreed on at the beginning, the simplex method will never terminate in this example.

Anticycling – Bland's Rule

We now discuss a pivoting rule that is guaranteed to avoid cycling:

Smallest subscript pivoting rule (Bland's rule)

1. Choose the column A_j with $\bar{c}_j < 0$ and j minimal to enter the basis.
2. Among all basic variables x_i that could exit the basis, select the one with smallest i.

Anticycling – Bland's Rule

We now discuss a pivoting rule that is guaranteed to avoid cycling:

Smallest subscript pivoting rule (Bland's rule)

1. Choose the column A_j with $\bar{c}_j < 0$ and j minimal to enter the basis.
2. Among all basic variables x_i that could exit the basis, select the one with smallest i.

Theorem (without proof)

The simplex algorithm with Bland's rule does not cycle and thus terminates after a finite number of iterations.

Finding an Initial Basic Feasible Solution

So far we always assumed that the simplex algorithm starts with a basic feasible solution. We now discuss how such a solution can be obtained.

- Introducing artificial variables
- The two-phase simplex method
- The big-M method

Introducing Artificial Variables

Example:

$$
\begin{array}{lrcrcrcrcl}
\min & x_1 & + & x_2 & + & x_3 & & & & \\
\text{s.t.} & x_1 & + & 2\,x_2 & + & 3\,x_3 & & & = & 3 \\
 & -x_1 & + & 2\,x_2 & + & 6\,x_3 & & & = & 2 \\
 & & & 4\,x_2 & + & 9\,x_3 & & & = & 5 \\
 & & & & & 3\,x_3 & + & x_4 & = & 1 \\
 & & & & & & & x_1,\ldots,x_4 & \geq & 0
\end{array}
$$

Introducing Artificial Variables

Example:

$$
\begin{array}{lrcrcrcrcl}
\min & x_1 & + & x_2 & + & x_3 & & & & \\
\text{s.t.} & x_1 & + & 2\,x_2 & + & 3\,x_3 & & & = & 3 \\
 & -x_1 & + & 2\,x_2 & + & 6\,x_3 & & & = & 2 \\
 & & & 4\,x_2 & + & 9\,x_3 & & & = & 5 \\
 & & & & & 3\,x_3 & + & x_4 & = & 1 \\
 & & & & & \multicolumn{3}{r}{x_1, \ldots, x_4} & \geq & 0
\end{array}
$$

Auxiliary problem with artificial variables:

$$
\begin{array}{lrrrrrrrrcl}
\min & & & & & x_5 & +x_6 & +x_7 & +x_8 & & \\
\text{s.t.} & x_1 & +2\,x_2 & +3\,x_3 & & +x_5 & & & & = & 3 \\
 & -x_1 & +2\,x_2 & +6\,x_3 & & & +x_6 & & & = & 2 \\
 & & 4\,x_2 & +9\,x_3 & & & & +x_7 & & = & 5 \\
 & & & 3\,x_3 & +x_4 & & & & +x_8 & = & 1 \\
 & & & & \multicolumn{5}{r}{x_1, \ldots, x_4, x_5, \ldots, x_8} & \geq & 0
\end{array}
$$

Auxiliary Problem

Auxiliary problem with artificial variables:

$$
\begin{array}{llllllllll}
\min & & & & & x_5 & +x_6 & +x_7 & +x_8 & & \\
\text{s.t.} & x_1 & +2\,x_2 & +3\,x_3 & & +x_5 & & & & = & 3 \\
 & -x_1 & +2\,x_2 & +6\,x_3 & & & +x_6 & & & = & 2 \\
 & & 4\,x_2 & +9\,x_3 & & & & +x_7 & & = & 5 \\
 & & & 3\,x_3 & +x_4 & & & & +x_8 & = & 1 \\
 & & & & & & & & x_1,\dots,x_4, x_5,\dots,x_8 & \geq & 0
\end{array}
$$

Auxiliary Problem

Auxiliary problem with artificial variables:

$$\begin{array}{lrrrrrrrrrl}
\min & & & & & x_5 & +x_6 & +x_7 & +x_8 & & \\
\text{s.t.} & x_1 & +2\,x_2 & +3\,x_3 & & +x_5 & & & & = & 3 \\
 & -x_1 & +2\,x_2 & +6\,x_3 & & & +x_6 & & & = & 2 \\
 & & 4\,x_2 & +9\,x_3 & & & & +x_7 & & = & 5 \\
 & & & 3\,x_3 & +x_4 & & & & +x_8 & = & 1 \\
 & & & & & & & & x_1,\ldots,x_4,x_5,\ldots,x_8 & \geq & 0
\end{array}$$

Observation

$x = (0,0,0,0,3,2,5,1)$ is a basic feasible solution for this problem with basic variables (x_5, x_6, x_7, x_8). We can form the initial tableau.

Initial Tableau

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	0	0	0	0	0	1	1	1	1
$x_5 =$	3	1	2	3	0	1	0	0	0
$x_6 =$	2	-1	2	6	0	0	1	0	0
$x_7 =$	5	0	4	9	0	0	0	1	0
$x_8 =$	1	0	0	3	1	0	0	0	1

Initial Tableau

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	0	0	0	0	0	1	1	1	1
$x_5 =$	3	1	2	3	0	1	0	0	0
$x_6 =$	2	-1	2	6	0	0	1	0	0
$x_7 =$	5	0	4	9	0	0	0	1	0
$x_8 =$	1	0	0	3	1	0	0	0	1

Calculate reduced costs by eliminating the nonzero-entries for the basis-variables.

Initial Tableau

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	-3	-1	-2	-3	0	0	1	1	1
$x_5 =$	3	1	2	3	0	1	0	0	0
$x_6 =$	2	-1	2	6	0	0	1	0	0
$x_7 =$	5	0	4	9	0	0	0	1	0
$x_8 =$	1	0	0	3	1	0	0	0	1

Calculate reduced costs by eliminating the nonzero-entries for the basis-variables.

Initial Tableau

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	-5	0	-4	-9	0	0	0	1	1
$x_5 =$	3	1	2	3	0	1	0	0	0
$x_6 =$	2	-1	2	6	0	0	1	0	0
$x_7 =$	5	0	4	9	0	0	0	1	0
$x_8 =$	1	0	0	3	1	0	0	0	1

Calculate reduced costs by eliminating the nonzero-entries for the basis-variables.

Initial Tableau

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	-10	0	-8	-18	0	0	0	0	1
$x_5 =$	3	1	2	3	0	1	0	0	0
$x_6 =$	2	-1	2	6	0	0	1	0	0
$x_7 =$	5	0	4	9	0	0	0	1	0
$x_8 =$	1	0	0	3	1	0	0	0	1

Calculate reduced costs by eliminating the nonzero-entries for the basis-variables.

Initial Tableau

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	-11	0	-8	-21	-1	0	0	0	0
$x_5 =$	3	1	2	3	0	1	0	0	0
$x_6 =$	2	-1	2	6	0	0	1	0	0
$x_7 =$	5	0	4	9	0	0	0	1	0
$x_8 =$	1	0	0	3	1	0	0	0	1

Calculate reduced costs by eliminating the nonzero-entries for the basis-variables.

Now we can proceed as seen before...

Minimizing the Auxiliary Problem

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	-11	0	-8	-21	-1	0	0	0	0
$x_5 =$	3	1	2	3	0	1	0	0	0
$x_6 =$	2	-1	2	6	0	0	1	0	0
$x_7 =$	5	0	4	9	0	0	0	1	0
$x_8 =$	1	0	0	3	1	0	0	0	1

Minimizing the Auxiliary Problem

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	-11	0	-8	-21	-1	0	0	0	0
$x_5 =$	3	1	2	3	0	1	0	0	0
$x_6 =$	2	-1	2	6	0	0	1	0	0
$x_7 =$	5	0	4	9	0	0	0	1	0
$x_8 =$	1	0	0	3	1	0	0	0	1

Basis change: x_4 enters the basis, x_8 exits.

Minimizing the Auxiliary Problem

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	-10	0	-8	-18	0	0	0	0	1
$x_5 =$	3	1	2	3	0	1	0	0	0
$x_6 =$	2	-1	2	6	0	0	1	0	0
$x_7 =$	5	0	4	9	0	0	0	1	0
$x_4 =$	1	0	0	3	1	0	0	0	1

Minimizing the Auxiliary Problem

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	-10	0	-8	-18	0	0	0	0	1
$x_5 =$	3	1	2	3	0	1	0	0	0
$x_6 =$	2	-1	2	6	0	0	1	0	0
$x_7 =$	5	0	4	9	0	0	0	1	0
$x_4 =$	1	0	0	3	1	0	0	0	1

Basis change: x_3 enters the basis, x_4 exits.

Minimizing the Auxiliary Problem

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	-4	0	-8	0	6	0	0	0	7
$x_5 =$	2	1	2	0	-1	1	0	0	-1
$x_6 =$	0	-1	2	0	-2	0	1	0	-2
$x_7 =$	2	0	4	0	-3	0	0	1	-3
$x_3 =$	$1/3$	0	0	1	$1/3$	0	0	0	$1/3$

Minimizing the Auxiliary Problem

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	-4	0	-8	0	6	0	0	0	7
$x_5 =$	2	1	2	0	-1	1	0	0	-1
$x_6 =$	0	-1	2	0	-2	0	1	0	-2
$x_7 =$	2	0	4	0	-3	0	0	1	-3
$x_3 =$	$1/3$	0	0	1	$1/3$	0	0	0	$1/3$

Basis change: x_2 enters the basis, x_6 exits.

Minimizing the Auxiliary Problem

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	-4	-4	0	0	-2	0	4	0	-1
$x_5 =$	2	2	0	0	1	1	-1	0	1
$x_2 =$	0	$-1/2$	1	0	-1	0	$1/2$	0	-1
$x_7 =$	2	2	0	0	1	0	-2	1	1
$x_3 =$	$1/3$	0	0	1	$1/3$	0	0	0	$1/3$

Minimizing the Auxiliary Problem

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	-4	-4	0	0	-2	0	4	0	-1
$x_5 =$	2	**2**	0	0	1	1	-1	0	1
$x_2 =$	0	$-1/2$	1	0	-1	0	$1/2$	0	-1
$x_7 =$	2	2	0	0	1	0	-2	1	1
$x_3 =$	$1/3$	0	0	1	$1/3$	0	0	0	$1/3$

Basis change: x_1 enters the basis, x_5 exits.

Minimizing the Auxiliary Problem

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	0	0	0	0	0	2	2	0	1
$x_1 =$	1	1	0	0	$1/2$	$1/2$	$-1/2$	0	$1/2$
$x_2 =$	$1/2$	0	1	0	$-3/4$	$1/4$	$1/4$	0	$-3/4$
$x_7 =$	0	0	0	0	0	-1	-1	1	0
$x_3 =$	$1/3$	0	0	1	$1/3$	0	0	0	$1/3$

Minimizing the Auxiliary Problem

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	0	0	0	0	0	2	2	0	1
$x_1 =$	1	1	0	0	$1/2$	$1/2$	$-1/2$	0	$1/2$
$x_2 =$	$1/2$	0	1	0	$-3/4$	$1/4$	$1/4$	0	$-3/4$
$x_7 =$	0	0	0	0	0	-1	-1	1	0
$x_3 =$	$1/3$	0	0	1	$1/3$	0	0	0	$1/3$

Basic feasible solution for auxiliary problem with (auxiliary) cost value 0

Minimizing the Auxiliary Problem

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	0	0	0	0	0	2	2	0	1
$x_1 =$	1	1	0	0	1/2	1/2	$-1/2$	0	1/2
$x_2 =$	1/2	0	1	0	$-3/4$	1/4	1/4	0	$-3/4$
$x_7 =$	0	0	0	0	0	-1	-1	1	0
$x_3 =$	1/3	0	0	1	1/3	0	0	0	1/3

Basic feasible solution for auxiliary problem with (auxiliary) cost value 0

$\Rightarrow$ Also feasible for the original problem - but not (yet) basic.

Obtaining a Basis for the Original Problem

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	0	0	0	0	0	2	2	0	1
$x_1 =$	1	1	0	0	$1/2$	$1/2$	$-1/2$	0	$1/2$
$x_2 =$	$1/2$	0	1	0	$-3/4$	$1/4$	$1/4$	0	$-3/4$
$x_7 =$	0	0	0	0	0	-1	-1	1	0
$x_3 =$	$1/3$	0	0	1	$1/3$	0	0	0	$1/3$

Obtaining a Basis for the Original Problem

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	0	0	0	0	0	2	2	0	1
$x_1 =$	1	1	0	0	$1/2$	$1/2$	$-1/2$	0	$1/2$
$x_2 =$	$1/2$	0	1	0	$-3/4$	$1/4$	$1/4$	0	$-3/4$
$x_7 =$	0	0	0	0	0	-1	-1	1	0
$x_3 =$	$1/3$	0	0	1	$1/3$	0	0	0	$1/3$

Observation

Restricting the tableau to the original variables, we get a zero-row.

Obtaining a Basis for the Original Problem

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	0	0	0	0	0	2	2	0	1
$x_1 =$	1	1	0	0	1/2	1/2	−1/2	0	1/2
$x_2 =$	1/2	0	1	0	−3/4	1/4	1/4	0	−3/4
$x_7 =$	0	0	0	0	0	−1	−1	1	0
$x_3 =$	1/3	0	0	1	1/3	0	0	0	1/3

Observation

Restricting the tableau to the original variables, we get a zero-row.
Thus the original equations are linearily dependent.

Obtaining a Basis for the Original Problem

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	0	0	0	0	0	2	2	0	1
$x_1 =$	1	1	0	0	1/2	1/2	−1/2	0	1/2
$x_2 =$	1/2	0	1	0	−3/4	1/4	1/4	0	−3/4
$x_7 =$	0	0	0	0	0	−1	−1	1	0
$x_3 =$	1/3	0	0	1	1/3	0	0	0	1/3

Observation

Restricting the tableau to the original variables, we get a zero-row.
Thus the original equations are linearily dependent.
$\rightarrow$ We can remove the third row.

Obtaining a Basis for the Original Problem

		x_1	x_2	x_3	x_4
	$*$	$*$	$*$	$*$	$*$
$x_1 =$	1	1	0	0	$1/2$
$x_2 =$	$1/2$	0	1	0	$-3/4$
$x_3 =$	$1/3$	0	0	1	$1/3$

Obtaining a Basis for the Original Problem

		x_1	x_2	x_3	x_4
	$*$	$*$	$*$	$*$	$*$
$x_1 =$	1	1	0	0	$1/2$
$x_2 =$	$1/2$	0	1	0	$-3/4$
$x_3 =$	$1/3$	0	0	1	$1/3$

We finally obtain a basic feasible solution for the original problem.

Obtaining a Basis for the Original Problem

		x_1	x_2	x_3	x_4
	0	1	1	1	0
$x_1 =$	1	1	0	0	$1/2$
$x_2 =$	$1/2$	0	1	0	$-3/4$
$x_3 =$	$1/3$	0	0	1	$1/3$

We finally obtain a basic feasible solution for the original problem.

Computing the reduced costs for this basis:

- Put original objective function in row 0.

Obtaining a Basis for the Original Problem

		x_1	x_2	x_3	x_4
	$-11/6$	0	0	0	$-1/12$
$x_1 =$	1	1	0	0	$1/2$
$x_2 =$	$1/2$	0	1	0	$-3/4$
$x_3 =$	$1/3$	0	0	1	$1/3$

We finally obtain a basic feasible solution for the original problem.

Computing the reduced costs for this basis:

- Put original objective function in row 0.
- Compute reduced costs by eliminating the nonzero entries for the basic variables.

Obtaining a Basis for the Original Problem

		x_1	x_2	x_3	x_4
	$-11/6$	0	0	0	$-1/12$
$x_1 =$	1	1	0	0	$1/2$
$x_2 =$	$1/2$	0	1	0	$-3/4$
$x_3 =$	$1/3$	0	0	1	$1/3$

We finally obtain a basic feasible solution for the original problem.

Computing the reduced costs for this basis:

- Put original objective function in row 0.
- Compute reduced costs by eliminating the nonzero entries for the basic variables.

The simplex method (phase II) can now start with its typical iterations.

Omitting Artificial Variables

Auxiliary problem

$$
\begin{array}{lrrrrrrrrrl}
\min & & & & & x_5 & +x_6 & +x_7 & +x_8 & & \\
\text{s.t.} & x_1 & +2\,x_2 & +3\,x_3 & & +x_5 & & & & = & 3 \\
& -x_1 & +2\,x_2 & +6\,x_3 & & & +x_6 & & & = & 2 \\
& & 4\,x_2 & +9\,x_3 & & & & +x_7 & & = & 5 \\
& & & 3\,x_3 & \color{red}{+x_4} & & & & \color{red}{+x_8} & = & \color{red}{1} \\
& & & & & & & \multicolumn{2}{r}{x_1,\ldots,x_8} & \geq & 0
\end{array}
$$

Artificial variable x_8 could have been omitted by setting x_4 to 1 in the initial basis. This is possible as x_4 does only appear in one constraint.

Omitting Artificial Variables

Auxiliary problem

$$
\begin{array}{lrrrrrrrrrl}
\min & & & & & x_5 & +x_6 & +x_7 & +x_8 & & \\
\text{s.t.} & x_1 & +2\,x_2 & +3\,x_3 & & +x_5 & & & & = & 3 \\
 & -x_1 & +2\,x_2 & +6\,x_3 & & & +x_6 & & & = & 2 \\
 & & 4\,x_2 & +9\,x_3 & & & & +x_7 & & = & 5 \\
 & & & 3\,x_3 & +x_4 & & & & +x_8 & = & 1 \\
 & & & & & & & & x_1, \ldots, x_8 & \geq & 0
\end{array}
$$

Artificial variable x_8 could have been omitted by setting x_4 to 1 in the initial basis. This is possible as x_4 does only appear in one constraint.

Generally, this can be done, e. g., with all slack variables that have nonnegative right hand sides.

Phase I of the Simplex Method

Given: LP in standard form: $\min\{c^T \cdot x \mid A \cdot x = b,\ x \geq 0\}$

Phase I of the Simplex Method

Given: LP in standard form: $\min\{c^T \cdot x \mid A \cdot x = b,\ x \geq 0\}$

1. Transform problem such that $b \geq 0$ (multiply constraints by -1).

Phase I of the Simplex Method

Given: LP in standard form: $\min\{c^T \cdot x \mid A \cdot x = b,\ x \geq 0\}$

1. Transform problem such that $b \geq 0$ (multiply constraints by -1).
2. Introduce artificial variables $y_1, \ldots, y_m$ and solve auxiliary problem

$$\min \sum_{i=1}^{m} y_i \quad \text{s.t. } A \cdot x + I_m \cdot y = b,\ x, y \geq 0\ .$$

Phase I of the Simplex Method

Given: LP in standard form: $\min\{c^T \cdot x \mid A \cdot x = b,\ x \geq 0\}$

1. Transform problem such that $b \geq 0$ (multiply constraints by -1).
2. Introduce artificial variables $y_1, \ldots, y_m$ and solve auxiliary problem

$$\min \sum_{i=1}^{m} y_i \quad \text{s.t. } A \cdot x + I_m \cdot y = b,\ x, y \geq 0 \ .$$

3. If optimal cost is positive, then STOP (original LP is infeasible).

Phase I of the Simplex Method

Given: LP in standard form: $\min\{c^T \cdot x \mid A \cdot x = b,\ x \geq 0\}$

1. Transform problem such that $b \geq 0$ (multiply constraints by -1).
2. Introduce artificial variables $y_1, \ldots, y_m$ and solve auxiliary problem
$$\min \sum_{i=1}^{m} y_i \quad \text{s.t. } A \cdot x + I_m \cdot y = b,\ x, y \geq 0 \ .$$
3. If optimal cost is positive, then STOP (original LP is infeasible).
4. If no artificial variable is in final basis, eliminate artificial variables and columns and STOP (feasible basis for original LP has been found).

Phase I of the Simplex Method

Given: LP in standard form: $\min\{c^T \cdot x \mid A \cdot x = b,\ x \geq 0\}$

1. Transform problem such that $b \geq 0$ (multiply constraints by -1).
2. Introduce artificial variables $y_1, \ldots, y_m$ and solve auxiliary problem

$$\min \sum_{i=1}^{m} y_i \quad \text{s.t. } A \cdot x + I_m \cdot y = b,\ x, y \geq 0 \ .$$

3. If optimal cost is positive, then STOP (original LP is infeasible).
4. If no artificial variable is in final basis, eliminate artificial variables and columns and STOP (feasible basis for original LP has been found).
5. If ℓth basic variable is artificial, find $j \in \{1, \ldots, n\}$ with ℓth entry in $B^{-1} \cdot A_j$ nonzero. Use this entry as pivot element and replace ℓth basic variable with x_j.

Phase I of the Simplex Method

Given: LP in standard form: $\min\{c^T \cdot x \mid A \cdot x = b,\ x \geq 0\}$

1. Transform problem such that $b \geq 0$ (multiply constraints by -1).
2. Introduce artificial variables $y_1, \dots, y_m$ and solve auxiliary problem
$$\min \sum_{i=1}^{m} y_i \quad \text{s.t. } A \cdot x + I_m \cdot y = b,\ x, y \geq 0 \ .$$
3. If optimal cost is positive, then STOP (original LP is infeasible).
4. If no artificial variable is in final basis, eliminate artificial variables and columns and STOP (feasible basis for original LP has been found).
5. If ℓth basic variable is artificial, find $j \in \{1, \dots, n\}$ with ℓth entry in $B^{-1} \cdot A_j$ nonzero. Use this entry as pivot element and replace ℓth basic variable with x_j.
6. If no such $j \in \{1, \dots, n\}$ exists, eliminate ℓth row (constraint).

The Two-phase Simplex Method

Two-phase simplex method

1. Given an LP in standard from, first run phase I.
2. If phase I yields a basic feasible solution for the original LP, enter "phase II" (see above).

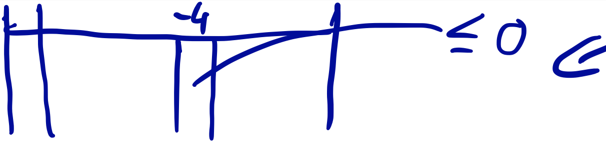
The Two-phase Simplex Method

Two-phase simplex method

1. Given an LP in standard from, first run phase I.
2. If phase I yields a basic feasible solution for the original LP, enter "phase II" (see above).

Possible outcomes of the two-phase simplex method

i. Problem is infeasible (detected in phase I).
ii. Problem is feasible but rows of A are linearly dependent (detected and corrected at the end of phase I by eliminating redundant constraints.)
iii. Optimal cost is $-\infty$ (detected in phase II).
iv. Problem has optimal basic feasible solution (found in phase II).



The Two-phase Simplex Method

Two-phase simplex method

1. Given an LP in standard from, first run phase I.
2. If phase I yields a basic feasible solution for the original LP, enter "phase II" (see above).

Possible outcomes of the two-phase simplex method

i. Problem is infeasible (detected in phase I).
ii. Problem is feasible but rows of A are linearly dependent (detected and corrected at the end of phase I by eliminating redundant constraints.)
iii. Optimal cost is $-\infty$ (detected in phase II).
iv. Problem has optimal basic feasible solution (found in phase II).

Remark: (ii) is not an outcome but only an intermediate result leading to outcome (iii) or (iv).