The Two-phase Simplex Method

Two-phase simplex method

1. Given an LP in standard from, first run phase I.
2. If phase I yields a basic feasible solution for the original LP, enter "phase II" (see above).

Possible outcomes of the two-phase simplex method

- i. Problem is infeasible (detected in phase I).
- ii. Problem is feasible but rows of A are linearly dependent (detected and corrected at the end of phase I by eliminating redundant constraints.)
- iii. Optimal cost is $-\infty$ (detected in phase II).
- iv. Problem has optimal basic feasible solution (found in phase II).

Remark: (ii) is not an outcome but only an intermediate result leading to outcome (iii) or (iv).

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becomes:

$$\begin{array}{lrcrcl} \min & \sum_{i=1}^{n} c_i x_i & + & M \cdot \sum_{j=1}^{m} y_j & & \\ \text{s.t.} & A \cdot x & + & I_m \cdot y & = & b \\ & & & x, y & \geq & 0 \end{array}$$

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Remark: If M is sufficiently large and the original program has a feasible solution, all artificial variables will be driven to zero by the simplex method.

How to Choose M?

Observation

Initially, M only occurs in the zeroth row. As the zeroth row never becomes pivot row, this property is maintained while the simplex method is running.

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$\rightarrow$ There is no need to give M a fixed numerical value.

Example

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$$
\begin{array}{lrcrcrcrcl}
\min & x_1 & + & x_2 & + & x_3 & & & & \\
\text{s.t.} & x_1 & + & 2\,x_2 & + & 3\,x_3 & & & = & 3 \\
 & -x_1 & + & 2\,x_2 & + & 6\,x_3 & & & = & 2 \\
 & & & 4\,x_2 & + & 9\,x_3 & & & = & 5 \\
 & & & & & 3\,x_3 & + & x_4 & = & 1 \\
 & & & & & & & x_1,\ldots,x_4 & \geq & 0
\end{array}
$$

Introducing Artificial Variables and M

make them zero

Auxiliary problem:

$$
\begin{array}{llllllllll}
\min & x_1 & +x_2 & +x_3 & & +M\,x_5 & +M\,x_6 & +M\,x_7 & & \\
\text{s.t.} & x_1 & +2\,x_2 & +3\,x_3 & & x_5 & & & = & 3 \\
& -x_1 & +2\,x_2 & +6\,x_3 & & & +x_6 & & = & 2 \\
& & 4\,x_2 & +9\,x_3 & & & & +x_7 & = & 5 \\
& & & 3\,x_3 & +x_4 & & & & = & 1 \\
& & & & & & & x_1,\ldots,x_4,x_5,x_6,x_7 & \geq & 0
\end{array}
$$

Note that this time the unnecessary artificial variable x_8 has been omitted.

We start off with $(x_5, x_6, x_7, x_4) = (3, 2, 5, 1)$.

Forming the Initial Tableau

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
0	1	1	1	0	M	M	M
3	1	2	3	0	1	0	0
2	−1	2	6	0	0	1	0
5	0	4	9	0	0	0	1
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Compute reduced costs by eliminating the nonzero entries for the basic variables.

Forming the Initial Tableau

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-3M$	$-M+1$	$-2M+1$	$-3M+1$	0	0	M	M
3	1	2	3	0	1	0	0
2	-1	2	6	0	0	1	0
5	0	4	9	0	0	0	1
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Compute reduced costs by eliminating the nonzero entries for the basic variables.

Forming the Initial Tableau

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-5M$	1	$-4M+1$	$-9M+1$	0	0	0	M
3	1	2	3	0	1	0	0
2	-1	2	6	0	0	1	0
5	0	4	9	0	0	0	1
1	0	0	3	1	0	0	0

Compute reduced costs by eliminating the nonzero entries for the basic variables.

Forming the Initial Tableau

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-10M$	1	$-8M+1$	$-18M+1$	0	0	0	0
3	1	2	3	0	1	0	0
2	-1	2	6	0	0	1	0
5	0	4	9	0	0	0	1
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Compute reduced costs by eliminating the nonzero entries for the basic variables.

First Iteration

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-10M$	1	$-8M+1$	$-18M+1$	0	0	0	0
3	1	2	3	0	1	0	0
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Reduced costs for x_2 and x_3 are negative.

First Iteration

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-10M$	1	$-8M+1$	$-18M+1$	0	0	0	0
3	1	2	3	0	1	0	0
2	-1	2	6	0	0	1	0
5	0	4	9	0	0	0	1
1	0	0	3	1	0	0	0

add $6M - \frac{1}{3}$ times

≤ 1, $\leq 1/3$, $\leq 5/9$, $x_3 \leq \frac{1}{3}$

Reduced costs for x_2 and x_3 are negative.

Basis change: x_3 enters the basis, x_4 leaves.

Second Iteration

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
$-4M-1/3$	1	$-8M+1$	0	$6M-1/3$	0	0	0	
2	1	2	0	-1	1	0	0	≤ 1
0	-1	2	0	-2	0	1	0	≤ 0
2	0	4	0	-3	0	0	1	$\leq \frac{1}{2}$
$1/3$	0	0	1	$1/3$	0	0	0	—

Second Iteration

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-4M - 1/3$	1	$-8M + 1$	0	$6M - 1/3$	0	0	0
2	1	2	0	-1	1	0	0
0	-1	2	0	-2	0	1	0
2	0	4	0	-3	0	0	1
$1/3$	0	0	1	$1/3$	0	0	0

Basis change: x_2 enters the basis, x_6 leaves.

Third Iteration

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-4M-1/3$	$-4M+3/2$	0	0	$-2M+2/3$	0	$4M-1/2$	0
2	2	0	0	1	1	-1	0
0	$-1/2$	1	0	-1	0	$1/2$	0
2	2	0	0	1	0	-2	1
$1/3$	0	0	1	$1/3$	0	0	0

Current solution:

$x_5 = 2$

$x_2 = 0$

$x_4 = 2$

$x_3 = \frac{1}{3}$

$\text{Obj} = 4M + \frac{1}{3}$

Third Iteration

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-4M - 1/3$	$-4M + 3/2$	0	0	$-2M + 2/3$	0	$4M - 1/2$	0
2	2	0	0	1	1	-1	0
0	$-1/2$	1	0	-1	0	$1/2$	0
2	2	0	0	1	0	-2	1
$1/3$	0	0	1	$1/3$	0	0	0

Basis change: x_1 enters the basis, x_5 leaves.

Fourth Iteration

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-11/6$	0	0	0	$-1/12$	$2M-3/4$	$2M+1/4$	0
21	1	0	0	$1/2$	$1/2$	$-1/2$	0
$1/2$	0	1	0	$-3/4$	$1/4$	$1/4$	0
0	0	0	0	0	-1	-1	1
$1/3$	0	0	1	$1/3$	0	0	0

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$1/2$	0	1	0	$-3/4$	$1/4$	$1/4$	0
0	0	0	0	0	-1	-1	1
$1/3$	0	0	1	$1/3$	0	0	0

Note that all artificial variables have already been driven to 0.

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$-11/6$	0	0	0	$-1/12$	$2M - 3/4$	$2M + 1/4$	0
21	1	0	0	$1/2$	$1/2$	$-1/2$	0
$1/2$	0	1	0	$-3/4$	$1/4$	$1/4$	0
0	0	0	0	0	-1	-1	1
$1/3$	0	0	1	$1/3$	0	0	0

≤ 42
—
—
≤ 1

Note that all artificial variables have already been driven to 0.

Basis change: x_4 enters the basis, x_3 leaves.

Fifth Iteration

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-7/4$	0	0	$1/4$	0	$2M - 3/4$	$2M + 1/4$	0
$1/2$	1	0	$-3/2$	0	$1/2$	$-1/2$	0
$5/4$	0	1	$9/4$	0	$1/4$	$1/4$	0
0	0	0	0	0	-1	-1	1
1	0	0	3	1	0	0	0

Fifth Iteration

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-7/4$	0	0	$1/4$	0	$2M - 3/4$	$2M + 1/4$	0
$1/2$	1	0	$-3/2$	0	$1/2$	$-1/2$	0
$5/4$	0	1	$9/4$	0	$1/4$	$1/4$	0
0	0	0	0	0	-1	-1	1
1	0	0	3	1	0	0	0

We now have an optimal solution of the auxiliary problem, as all costs are nonnegative (M presumed large enough).

Fifth Iteration

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-7/4$	0	0	$1/4$	0	$2M-3/4$	$2M+1/4$	0
$1/2$	1	0	$-3/2$	0	$1/2$	$-1/2$	0
$5/4$	0	1	$9/4$	0	$1/4$	$1/4$	0
0	0	0	0	0	-1	-1	1
1	0	0	3	1	0	0	0

We now have an optimal solution of the auxiliary problem, as all costs are nonnegative (M presumed large enough).

By elimiating the third row as in the previous example, we get a basic feasible and also optimal solution to the original problem.

$x_1 = \frac{1}{2}$ $\quad$ $x_i = 0$ otherwise

$x_2 = \frac{5}{4}$

$x_7 = 0$ $\quad\quad$ opt $= \frac{4}{4}$

$x_4 = 1$

Computational Efficiency of the Simplex Method

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The computational efficiency of the simplex method is determined by

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Idea for negative answer (lower bound)

Describe

- a polyhedron with an exponential number of vertices;
- a path that visits all vertices and always moves from a vertex to an adjacent one that has lower costs.

Computational Efficiency of the Simplex Method

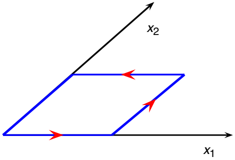
Unit cube

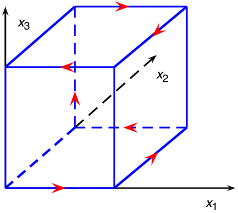
Consider the unit cube in $\mathbb{R}^n$, defined by the constraints

$$0 \le x_i \le 1, \quad i = 1, \ldots, n$$

The unit cube has

- 2^n vertices;
- a *spanning path*, i. e., a path traveling the edges of the cube visiting each vertex exactly once.





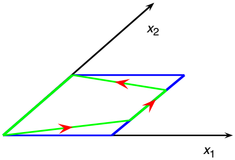
Computational Efficiency of the Simplex Method (cont.)

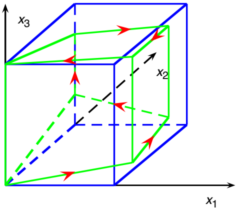
Klee-Minty cube

Consider a perturbation of the unit cube in $\mathbb{R}^n$, defined by the constraints

$$0 \leq x_1 \leq 1,$$
$$\epsilon x_{i-1} \leq x_i \leq 1 - \epsilon x_{i-1}, \quad i = 2, \ldots, n$$

for some $\epsilon \in (0, 1/2)$.





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Computational Efficiency of the Simplex Method (cont.)

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Theorem 3.1.

Consider the linear programming problem of minimizing $-x_n$ subject to the constraints above. Then,

a. the feasible set has 2^n vertices;
b. the vertices can be ordered so that each one is adjacent to and has lower cost than the previous one;
c. there exists a pivoting rule under which the simplex method requires $2^n - 1$ changes of basis before it terminates.

Diameter of Polyhedra

Definition 3.2.

- The distance $d(x, y)$ between two vertices x, y is the minimum number of edges required to reach y starting from x.
- The diameter $D(P)$ of polyhedron P is the maximum $d(x, y)$ over all pairs of vertices (x, y).
- $\Delta(n, m)$ is the maximum $D(P)$ over all polytopes in $\mathbb{R}^n$ that are represented in terms of m inequality constraints.
- $\Delta_u(n, m)$ is the maximum $D(P)$ over all polyhedra in $\mathbb{R}^n$ that are represented in terms of m inequality constraints.

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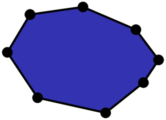

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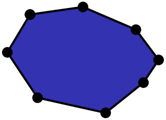

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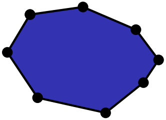

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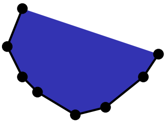

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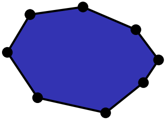

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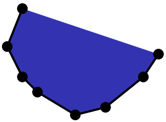

$$\Delta_u(2, 8) = 8 - 2 = 6$$
$$\Delta_u(2, m) = m - 2$$

Hirsch Conjecture

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Polynomial Hirsch Conjecture

$$\Delta(n, m) \leq poly(m, n)$$

Hirsch Conjecture

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Polynomial Hirsch Conjecture

$$\Delta(n, m) \leq poly(m, n)$$

Remarks

- Known lower bounds: $\Delta_u(n, m) \geq m - n + \left\lfloor \frac{n}{5} \right\rfloor$

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- The Strong Hirsch Conjecture

$$\Delta(n, m) \leq m - n$$

was disproven in 2010 by Paco Santos for $n = 43$, $m = 86$.

Average Case Behavior of the Simplex Method

- Despite the exponential lower bounds on the worst case behavior of the simplex method (Klee-Minty cubes etc.), the simplex method usually behaves well in practice.
- The number of iterations is “typically” $O(m)$.
- There have been several attempts to explain this phenomenon from a more theoretical point of view.
- These results say that “on average” the number of iterations is $O(\cdot)$ (usually polynomial).
- One main difficulty is to come up with a meaningful and, at the same time, manageable definition of the term “on average”.