COMP331/557

Chapter 4:
Duality Theory

(Bertsimas & Tsitsiklis, Chapter 4)

Example

$$\begin{array}{rrcrcl}
\text{minimize} & x_1 & + & 2x_2 & & \\
\text{s.t.} & x_1 & & & \geq & 2 \\
& & & x_2 & \geq & 2 \\
& -x_1 & + & x_2 & \geq & 1 \\
& x_1 & + & x_2 & \geq & 5 \\
& & & x_1, x_2 & \geq & 0
\end{array}$$

Goal: Find an upper bound on the optimal solution value z^*.

Example

$$\begin{array}{rrcrcr} \text{minimize} & x_1 & + & 2x_2 & & \\ \text{s.t.} & x_1 & & & \geq & 2 \\ & & & x_2 & \geq & 2 \\ & -x_1 & + & x_2 & \geq & 1 \\ & x_1 & + & x_2 & \geq & 5 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

Goal: Find an upper bound on the optimal solution value z^*.

Easy: Any feasible solution provides one.

Example

$$\begin{array}{rrcrcl} \text{minimize} & x_1 & + & 2x_2 & & \\ \text{s.t.} & x_1 & & & \geq & 2 \\ & & & x_2 & \geq & 2 \\ & -x_1 & + & x_2 & \geq & 1 \\ & x_1 & + & x_2 & \geq & 5 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

Goal: Find an upper bound on the optimal solution value z^*.

Easy: Any feasible solution provides one.

Examples:

- $(x_1, x_2) = (4, 5) \qquad \Rightarrow z^* \leq 14$

Example

$$\begin{array}{rrcrcr} \text{minimize} & x_1 & + & 2x_2 & & \\ \text{s.t.} & x_1 & & & \geq & 2 \\ & & & x_2 & \geq & 2 \\ & -x_1 & + & x_2 & \geq & 1 \\ & x_1 & + & x_2 & \geq & 5 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

Goal: Find an upper bound on the optimal solution value z^*.

Easy: Any feasible solution provides one.

Examples:

- $(x_1, x_2) = (4, 5) \quad \Rightarrow z^* \leq 14$
- $(x_1, x_2) = (3, 4) \quad \Rightarrow z^* \leq 11$

Example

$$\begin{array}{rrcrcl} \text{minimize} & x_1 & + & 2x_2 & & \\ \text{s.t.} & x_1 & & & \geq & 2 \\ & & & x_2 & \geq & 2 \\ & -x_1 & + & x_2 & \geq & 1 \\ & x_1 & + & x_2 & \geq & 5 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

Goal: Find an upper bound on the optimal solution value z^*.

Easy: Any feasible solution provides one.

Examples:

- $(x_1, x_2) = (4, 5) \qquad \Rightarrow z^* \leq 14$
- $(x_1, x_2) = (3, 4) \qquad \Rightarrow z^* \leq 11$
- $(x_1, x_2) = (2, 4) \qquad \Rightarrow z^* \leq 10$

Example

$$
\begin{array}{rrcrcl}
\text{minimize} & x_1 & + & 2x_2 & & \\
\text{s.t.} & x_1 & & & \geq & 2 \\
 & & & x_2 & \geq & 2 \\
 & -x_1 & + & x_2 & \geq & 1 \\
 & x_1 & + & x_2 & \geq & 5 \\
 & & & x_1, x_2 & \geq & 0
\end{array}
$$

Goal: Find an upper bound on the optimal solution value z^*.

Easy: Any feasible solution provides one.

Examples:

- $(x_1, x_2) = (4, 5) \quad \Rightarrow z^* \leq 14$
- $(x_1, x_2) = (3, 4) \quad \Rightarrow z^* \leq 11$
- $(x_1, x_2) = (2, 4) \quad \Rightarrow z^* \leq 10$
- $(x_1, x_2) = (2, 3) \quad \Rightarrow z^* \leq 8$

Example

$$
\begin{array}{rrcrcl@{\qquad}l}
\text{minimize} & x_1 & + & 2x_2 & & & \leftarrow z \\
\text{s.t.} & x_1 & & & \geq & 2 & \leftarrow C_1 \\
 & & & x_2 & \geq & 2 & \leftarrow C_2 \\
 & -x_1 & + & x_2 & \geq & 1 & \leftarrow C_3 \\
 & x_1 & + & x_2 & \geq & 5 & \leftarrow C_4 \\
 & & & x_1, x_2 & \geq & 0 &
\end{array}
$$

New goal: Find a lower bound on the optimal solution value.

Example

$$
\begin{array}{rrcrcll}
\text{minimize} & x_1 & + & 2x_2 & & & \leftarrow z \\
\text{s.t.} & x_1 & & & \geq & 2 & \leftarrow C_1 \\
 & & & x_2 & \geq & 2 & \leftarrow C_2 \\
 & -x_1 & + & x_2 & \geq & 1 & \leftarrow C_3 \\
 & x_1 & + & x_2 & \geq & 5 & \leftarrow C_4 \\
 & & & x_1, x_2 & \geq & 0 &
\end{array}
$$

New goal: Find a lower bound on the optimal solution value.

Examples:

- C_4

Example

$$\begin{array}{rrcrcrl}
\text{minimize} & x_1 & + & 2x_2 & & & \leftarrow z \\
\text{s.t.} & x_1 & & & \geq & 2 & \leftarrow C_1 \\
 & & & x_2 & \geq & 2 & \leftarrow C_2 \\
 & -x_1 & + & x_2 & \geq & 1 & \leftarrow C_3 \\
 & x_1 & + & x_2 & \geq & 5 & \leftarrow C_4 \\
 & & & x_1, x_2 & \geq & 0 &
\end{array}$$

New goal: Find a lower bound on the optimal solution value.

Examples:

- $C_4 \quad \Rightarrow \quad z \geq 5$

Example

$$\begin{array}{rrcrcll}
\text{minimize} & x_1 & + & 2x_2 & & & \leftarrow z \\
\text{s.t.} & x_1 & & & \geq & 2 & \leftarrow C_1 \\
 & & & x_2 & \geq & 2 & \leftarrow C_2 \\
 & -x_1 & + & x_2 & \geq & 1 & \leftarrow C_3 \\
 & x_1 & + & x_2 & \geq & 5 & \leftarrow C_4 \\
 & & & x_1, x_2 & \geq & 0 &
\end{array}$$

New goal: Find a lower bound on the optimal solution value.

Examples:

- $C_4 \quad \Rightarrow \quad z \geq 5$
- $C_1 + 2\ C_2 \quad \Rightarrow \quad x_1 + 2x_2 \geq 6 \quad \Rightarrow \quad z \geq 6$

Example

$$\begin{array}{rrcrcll}
\text{minimize} & x_1 & + & 2x_2 & & & \leftarrow z \\
\text{s.t.} & x_1 & & & \geq & 2 & \leftarrow C_1 \\
 & & & x_2 & \geq & 2 & \leftarrow C_2 \\
 & -x_1 & + & x_2 & \geq & 1 & \leftarrow C_3 \\
 & x_1 & + & x_2 & \geq & 5 & \leftarrow C_4 \\
 & & & x_1, x_2 & \geq & 0 &
\end{array}$$

New goal: Find a lower bound on the optimal solution value.

Examples:

- $C_4 \quad \Rightarrow \quad z \geq 5$
- $C_1 + 2\ C_2 \quad \Rightarrow \quad z \geq 6$

Example

$$
\begin{array}{rrcrcrl}
\text{minimize} & x_1 & + & 2x_2 & & & \leftarrow z \\
\text{s.t.} & x_1 & & & \geq & 2 & \leftarrow C_1 \\
 & & & x_2 & \geq & 2 & \leftarrow C_2 \\
 & -x_1 & + & x_2 & \geq & 1 & \leftarrow C_3 \\
 & x_1 & + & x_2 & \geq & 5 & \leftarrow C_4 \\
 & & & x_1, x_2 & \geq & 0 &
\end{array}
$$

New goal: Find a lower bound on the optimal solution value.

Examples:

- $C_4 \quad \Rightarrow \quad z \geq 5$
- $C_1 + 2\,C_2 \quad \Rightarrow \quad z \geq 6$
- $3\,C_1 + 2\,C_3 \quad \Rightarrow \quad 3x_1 - 2x_1 + 2x_2 \geq 3 \cdot 2 + 2 = 8 \Rightarrow z \geq 8$
- $3\,C_2 - C_3 \quad \Rightarrow$

$3C_2 : \quad 3x_2 \boxed{\geq} 2$

$-C_3 : \quad x_1 - x_2 \boxed{\leq} -1$

→ does not give lower bound

Example

$$\begin{array}{rrcrcrl}
\text{minimize} & x_1 & + & 2x_2 & & & \leftarrow z \\
\text{s.t.} & x_1 & & & \geq & 2 & \leftarrow C_1 \\
 & & & x_2 & \geq & 2 & \leftarrow C_2 \\
 & -x_1 & + & x_2 & \geq & 1 & \leftarrow C_3 \\
 & x_1 & + & x_2 & \geq & 5 & \leftarrow C_4 \\
 & & & x_1, x_2 & \geq & 0 &
\end{array}$$

Idea: Add non-negative combination $p_1 \cdot C_1 + p_2 \cdot C_2 + p_3 \cdot C_3 + p_4 \cdot C_4$ of the constraints, s.t.:

Example

$$
\begin{array}{rrcrcll}
\text{minimize} & x_1 & + & 2x_2 & & & \leftarrow z \\
\text{s.t.} & x_1 & & & \geq & 2 & \leftarrow C_1 \quad \cdot\, p_1 \\
 & & & x_2 & \geq & 2 & \leftarrow C_2 \quad \cdot\, p_2 \\
 & -x_1 & + & x_2 & \geq & 1 & \leftarrow C_3 \quad \cdot\, p_3 \\
 & x_1 & + & x_2 & \geq & 5 & \leftarrow C_4 \quad \cdot\, p_4 \\
 & & & x_1, x_2 & \geq & 0 &
\end{array}
$$

Idea: Add non-negative combination $p_1 \cdot C_1 + p_2 \cdot C_2 + p_3 \cdot C_3 + p_4 \cdot C_4$ of the constraints, s.t.:

$$
\begin{aligned}
z = x_1 + 2x_2 &\geq (p_1 - p_3 + p_4) \cdot x_1 + (p_2 + p_3 + p_4) \cdot x_2 \\
&\geq 2\, p_1 + 2\, p_2 + p_3 + 5\, p_4
\end{aligned}
$$

Example

$$\begin{array}{rrcrcll} \text{minimize} & x_1 & + & 2x_2 & & & \leftarrow z \\ \text{s.t.} & x_1 & & & \geq & 2 & \leftarrow C_1 \\ & & & x_2 & \geq & 2 & \leftarrow C_2 \\ & -x_1 & + & x_2 & \geq & 1 & \leftarrow C_3 \\ & x_1 & + & x_2 & \geq & 5 & \leftarrow C_4 \\ & & & x_1, x_2 & \geq & 0 & \end{array}$$

Idea: Add non-negative combination $p_1 \cdot C_1 + p_2 \cdot C_2 + p_3 \cdot C_3 + p_4 \cdot C_4$ of the constraints, s.t.:

$$\begin{aligned} z = x_1 + 2x_2 &\geq (p_1 - p_3 + p_4) \cdot x_1 + (p_2 + p_3 + p_4) \cdot x_2 \\ &\geq \underline{2\ p_1 + 2\ p_2 + p_3 + 5\ p_4} \end{aligned}$$

Dual Problem:
Find the best such lower bound.

$$\begin{array}{ll} \max & 2p_1 + 2p_2 + p_3 + 5p_4 \\ \text{s.t.} & p_1 - p_3 + p_4 \leq 1 \\ & p_2 + p_3 + p_4 \leq 2 \\ & p_1, p_2, p_3, p_4 \geq 0 \end{array}$$

More general

$$\begin{array}{rcccccccll}
\text{minimize} & c_1x_1 & + & \cdots & + & c_nx_n & & & \leftarrow z & \\
\text{s.t.} & a_{11}x_1 & + & \cdots & + & a_{1n}x_n & \geq & b_1 & \leftarrow C_1 & p_1 \\
 & a_{21}x_1 & + & \cdots & + & a_{2n}x_n & \geq & b_2 & \leftarrow C_2 & p_2 \\
 & \vdots & & \ddots & & & \vdots & & & \vdots \\
 & a_{m1}x_1 & + & \cdots & + & a_{mn}x_n & \geq & b_m & \leftarrow C_m & p_m \\
 & & & & & x_1, \ldots, x_n & \geq & 0 & &
\end{array}$$

Consider: $p_1 C_1 + p_2 C_2 + \cdots + p_m C_m$

More general

$$\begin{array}{rccccccccl}
\text{minimize} & c_1x_1 & + & \cdots & + & c_nx_n & & & \leftarrow z & \\
\text{s.t.} & a_{11}x_1 & + & \cdots & + & a_{1n}x_n & \geq & b_1 & \leftarrow C_1 & \cdot p_1 \\
 & a_{21}x_1 & + & \cdots & + & a_{2n}x_n & \geq & b_2 & \leftarrow C_2 & \cdot p_2 \\
 & \vdots & & \ddots & & & \vdots & & & \\
 & a_{m1}x_1 & + & \cdots & + & a_{mn}x_n & \geq & b_m & \leftarrow C_m & \cdot p_m \\
 & & & & & x_1, \ldots, x_n & \geq & 0 & &
\end{array}$$

Consider: $p_1 C_1 + p_2 C_2 + \cdots + p_m C_m$

Q: What are the conditions on $p_1, \ldots, p_m$ so that this combination lower bounds z?

More general

$$\begin{array}{rccccccccl}
\text{minimize} & c_1x_1 & + & \cdots & + & c_nx_n & & & & \leftarrow z \\
\text{s.t.} & a_{11}x_1 & + & \cdots & + & a_{1n}x_n & \geq & b_1 & & \leftarrow C_1 \\
 & a_{21}x_1 & + & \cdots & + & a_{2n}x_n & \geq & b_2 & & \leftarrow C_2 \\
 & \vdots & & \ddots & & & \vdots & & & \\
 & a_{m1}x_1 & + & \cdots & + & a_{mn}x_n & \geq & b_m & & \leftarrow C_m \\
 & & & & & x_1, \dots, x_n & \geq & 0 & &
\end{array}$$

Consider: $p_1 C_1 + p_2 C_2 + \cdots + p_m C_m$

Q: What are the conditions on $p_1, \dots, p_m$ so that this combination lower bounds z?

$$\begin{array}{rcl}
a_{11}p_1 + a_{21}p_2 + \cdots + a_{m1}p_m & \leq & c_1 \\
\vdots \qquad \vdots \qquad \vdots & \leq & \vdots \\
a_{1n}p_1 + a_{1n}p_2 + \cdots + a_{mn}p_m & \leq & c_n \\
p_1, p_2, \dots, p_m & \geq & 0
\end{array}$$

More general

$$\begin{array}{rccccccl}
\text{minimize} & c_1 x_1 & + & \cdots & + & c_n x_n & & & \leftarrow z \\
\text{s.t.} & a_{11} x_1 & + & \cdots & + & a_{1n} x_n & \geq & b_1 & \leftarrow C_1 \\
 & a_{21} x_1 & + & \cdots & + & a_{2n} x_n & \geq & b_2 & \leftarrow C_2 \\
 & \vdots & & \ddots & & & \vdots & & \\
 & a_{m1} x_1 & + & \cdots & + & a_{mn} x_n & \geq & b_m & \leftarrow C_m \\
 & & & & & x_1, \ldots, x_n & \geq & 0 &
\end{array}$$

Consider: $p_1 C_1 + p_2 C_2 + \cdots + p_m C_m$

Q: What are the conditions on $p_1, \ldots, p_m$ so that this combination lower bounds z?

$$\begin{array}{rcl}
a_{11} p_1 + a_{21} p_2 + \cdots + a_{m1} p_m & \leq & c_1 \\
\vdots \quad \vdots \quad \vdots & \leq & \vdots \\
a_{1n} p_1 + a_{1n} p_2 + \cdots + a_{mn} p_m & \leq & c_n \\
p_1, p_2, \ldots, p_m & \geq & 0
\end{array}$$

Q: What lower bound do we get?

$p_1 b_1 + p_2 b_2 + \cdots + p_m b_m$

Primal and Dual LP

Primal: Decision variables $x_1, \dots, x_n$.

$$\begin{array}{rccccccc} \text{minimize} & c_1x_1 & + & \cdots & + & c_nx_n & & \\ \text{s.t.} & a_{11}x_1 & + & \cdots & + & a_{1n}x_n & \geq & b_1 \\ & a_{21}x_1 & + & \cdots & + & a_{2n}x_n & \geq & b_2 \\ & \vdots & & \ddots & & & \vdots & \\ & a_{m1}x_1 & + & \cdots & + & a_{mn}x_n & \geq & b_m \\ & & & & & x_1, \dots, x_n & \geq & 0 \end{array}$$

$$\begin{array}{rll} \min & c^T x & \\ \text{s.t.} & A\,x & \geq b \\ & x & \geq 0 \end{array}$$

Dual: Decision variables $p_1, \dots, p_m$.

$$\begin{array}{rccccccc} \text{maximize} & b_1p_1 & + & \cdots & + & b_mp_m & & \\ \text{s.t.} & a_{11}p_1 & + & \cdots & + & a_{m1}p_m & \leq & c_1 \\ & a_{12}p_1 & + & \cdots & + & a_{m2}p_m & \leq & c_2 \\ & \vdots & & \ddots & & & \vdots & \\ & a_{1n}p_1 & + & \cdots & + & a_{mn}p_m & \leq & c_n \\ & & & & & p_1, \dots, p_m & \geq & 0 \end{array}$$

$$\begin{array}{rll} \max & b^T p & \\ \text{s.t.} & A^T p & \leq c \\ & p & \geq 0 \end{array}$$

Primal and Dual Example (1)

Primal:

$$\begin{array}{lrcrcrl}
\min & x_1 & + & 2x_2 & & & \\
\text{s.t.} & 2x_1 & + & x_2 & \geq & 7 & \Leftarrow p_1 \\
 & -x_1 & + & 3x_2 & \geq & 1 & \Leftarrow p_2 \\
 & x_1 & + & 4x_2 & \geq & 5 & \Leftarrow p_3 \\
 & & & x_1, x_2 & \geq & 0 &
\end{array}$$

Dual:

$$\begin{array}{ll}
\max & 7p_1 + p_2 + 5p_3 \\
 & 2p_1 - p_2 + p_3 \leq 1 \\
 & p_1 + 3p_2 + 4p_3 \leq 2 \\
 & p_j \geq 0 \quad \forall j
\end{array}$$

Primal and Dual Example (2)

Primal:

$$\begin{array}{lrcrcl} \min & -x_1 & + & 4x_2 & & \\ \text{s.t.} & 3x_1 & + & 2x_2 & \geq & 9 \quad \leftarrow p_1 \\ & x_1 & - & 3x_2 & \leq & 3 \quad \Longleftrightarrow \quad -x_1 + 3x_2 \geq -3 \quad \leftarrow p_2' \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

Dual:

$$p_2 := -p_2'$$

$$\begin{array}{ll} \max & 9p_1 - 3p_2' \\ \text{s.t.} & 3p_1 - p_2' \leq -1 \\ & 2p_1 + 3p_2' \leq 4 \\ & p_1 \geq 0 \\ & p_2' \geq 0 \end{array} \quad \Longleftrightarrow \quad \begin{array}{ll} \max & 9p_1 + 3p_2 \\ \text{s.t.} & 3p_1 + p_2 \leq -1 \\ & 2p_1 - 3p_2 \leq 1 \\ & p_1 \geq 0 \\ & p_2 \leq 0 \end{array}$$

Primal and Dual Example (3)

Primal:

$$\begin{array}{lrcrcl}
\min & -x_1 & + & 4x_2 & & \\
\text{s.t.} & 3x_1 & + & 2x_2 & \geq & 9 \\
 & x_1 & - & 3x_2 & = & 3 \\
 & & & x_1, x_2 & \geq & 0
\end{array}$$

p_1

$$\left\{\begin{array}{l} x_1 - 3x_2 \geq 3 \leftarrow p_2' \\ -x + 3x_2 \geq -3 \leftarrow p_2'' \end{array}\right.$$

Dual:

$$\begin{array}{ll}
\max & 9p_1 + 3(p_2' - p_2'') \\
\text{s.t} & 3p_1 + (p_2' - p_2'') \leq -1 \\
 & 2p_1 - 3(p_2' - p_2'') \leq 4 \\
 & p_1 \geq 0 \\
 & p_2' \geq 0 \\
 & p_2'' \geq 0
\end{array}$$

$$p_2 := p_2' - p_2'' \quad \Longleftrightarrow$$

$$\begin{array}{ll}
\max & 9p_1 + 3p_2 \\
\text{s.t.} & 3p_1 + p_2 \leq -1 \\
 & 2p_1 - 3p_2 \leq 4 \\
 & p_1 \geq 0 \\
 & p_2 \text{ free}
\end{array}$$