Primal and Dual Example (1)

Primal:

$$
\begin{array}{lrcrcl}
\min & x_1 & + & 2x_2 & & \\
\text{s.t.} & 2x_1 & + & x_2 & \geq & 7 \quad \Leftarrow p_1 \\
 & -x_1 & + & 3x_2 & \geq & 1 \quad \Leftarrow p_2 \\
 & x_1 & + & 4x_2 & \geq & 5 \quad \Leftarrow p_3 \\
 & & & x_1, x_2 & \geq & 0
\end{array}
$$

Dual:

$$
\begin{array}{ll}
\max & 7p_1 + p_2 + 5p_3 \\
 & 2p_1 - p_2 + p_3 \leq 1 \\
 & p_1 + 3p_2 + 4p_3 \leq 2 \\
 & p_j \geq 0 \quad \forall j
\end{array}
$$

Primal and Dual Example (2)

Primal:

$$\begin{array}{lrcrcl} \min & -x_1 & + & 4x_2 & & \\ \text{s.t.} & 3x_1 & + & 2x_2 & \geq & 9 \\ & x_1 & - & 3x_2 & \leq & 3 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

p_1

$\Leftrightarrow \quad -x_1 + 3x_2 \geq -3 \quad p_2'$

Dual:

$$\begin{array}{ll} \max & 9p_1 - 3p_2' \\ \text{s.t.} & 3p_1 - p_2' \leq -1 \\ & 2p_1 + 3p_2' \leq 4 \\ & p_1 \geq 0 \\ & p_2' \geq 0 \end{array}$$

$p_2 := -p_2'$

$\Longleftrightarrow$

$$\begin{array}{ll} \max & 9p_1 + 3p_2 \\ \text{s.t.} & 3p_1 + p_2 \leq -1 \\ & 2p_1 - 3p_2 \leq \\ & p_1 \geq 0 \\ & p_2 \leq 0 \end{array}$$

Primal and Dual Example (3)

Primal:

$$
\begin{array}{lrcrcr}
\min & -x_1 & + & 4x_2 & & \\
\text{s.t.} & 3x_1 & + & 2x_2 & \geq & 9 \\
 & x_1 & - & 3x_2 & = & 3 \\
 & & & x_1, x_2 & \geq & 0
\end{array}
$$

p_1

$$
\begin{cases} x_1 - 3x_2 \geq 3 & \leftarrow p_2' \\ -x_1 + 3x_2 \geq -3 & \leftarrow p_2'' \end{cases}
$$

Dual:

$$
\begin{array}{ll}
\max & 9p_1 + 3(p_2' - p_2'') \\
\text{s.t.} & 3p_1 + (p_2' - p_2'') \leq -1 \\
 & 2p_1 - 3(p_2' - p_2'') \leq 4 \\
 & p_1 \geq 0 \\
 & p_2' \geq 0 \\
 & p_2'' \geq 0
\end{array}
$$

$p_2 := p_2' - p_2''$

$\Longleftrightarrow$

$$
\begin{array}{ll}
\max & 9p_1 + 3p_2 \\
\text{s.t.} & 3p_1 + p_2 \leq -1 \\
 & 2p_1 - 3p_2 \leq 4 \\
 & p_1 \geq 0 \\
 & p_2 \text{ free}
\end{array}
$$

Primal and Dual Linear Program

Consider the general linear program:

$$
\begin{array}{llll}
\min & c^T \cdot x & & \\
\text{s.t.} & {a_i}^T \cdot x \geq b_i & \text{for } i \in M_1 \\
& {a_i}^T \cdot x \leq b_i & \text{for } i \in M_2 \\
& {a_i}^T \cdot x = b_i & \text{for } i \in M_3 \\
& x_j \geq 0 & \text{for } j \in N_1 \\
& x_j \leq 0 & \text{for } j \in N_2 \\
& x_j \text{ free} & \text{for } j \in N_3
\end{array}
$$

$p_i \geq 0$ for $i \in M_1$
$p_i \leq 0$ for $i \in M_2$
p_i free for $i \in M_3$

Primal and Dual Linear Program

Consider the general linear program:

$$
\begin{array}{lll}
\min & c^T \cdot x & \\
\text{s.t.} & a_i^T \cdot x \geq b_i & \text{for } i \in M_1 \\
& a_i^T \cdot x \leq b_i & \text{for } i \in M_2 \\
& a_i^T \cdot x = b_i & \text{for } i \in M_3 \\
& x_j \geq 0 & \text{for } j \in N_1 \\
& x_j \leq 0 & \text{for } j \in N_2 \\
& x_j \text{ free} & \text{for } j \in N_3
\end{array}
$$

Obtain a lower bound:

$$
\begin{array}{ll}
p_i \geq 0 & \text{for } i \in M_1 \\
p_i \leq 0 & \text{for } i \in M_2 \\
p_i \text{ free} & \text{for } i \in M_3 \\
A_j^T \cdot p \leq c_j & \text{for } j \in N_1 \\
A_j^T \cdot p \geq c_j & \text{for } j \in N_2 \\
A_j^T \cdot p = c_j & \text{for } j \in N_3
\end{array}
$$

Primal and Dual Linear Program

Consider the general linear program:

$$
\begin{array}{llll}
\min & c^T \cdot x & & \\
\text{s.t.} & {a_i}^T \cdot x \geq b_i & & \text{for } i \in M_1 \\
& {a_i}^T \cdot x \leq b_i & & \text{for } i \in M_2 \\
& {a_i}^T \cdot x = b_i & & \text{for } i \in M_3 \\
& x_j \geq 0 & & \text{for } j \in N_1 \\
& x_j \leq 0 & & \text{for } j \in N_2 \\
& x_j \text{ free} & & \text{for } j \in N_3
\end{array}
$$

Obtain a lower bound:

$$
\begin{array}{llll}
\max & p^T b & & \\
\text{s.t.} & p_i \geq 0 & & \text{for } i \in M_1 \\
& p_i \leq 0 & & \text{for } i \in M_2 \\
& p_i \text{ free} & & \text{for } i \in M_3 \\
& A_j^T \cdot p \leq c_j & & \text{for } j \in N_1 \\
& A_j^T \cdot p \geq c_j & & \text{for } j \in N_2 \\
& A_j^T \cdot p = c_j & & \text{for } j \in N_3
\end{array}
$$

Primal and Dual Linear Program

Consider the general linear program:

$$
\begin{array}{lll}
\min & c^T \cdot x & \\
\text{s.t.} & a_i{}^T \cdot x \geq b_i & \text{for } i \in M_1 \\
& a_i{}^T \cdot x \leq b_i & \text{for } i \in M_2 \\
& a_i{}^T \cdot x = b_i & \text{for } i \in M_3 \\
& x_j \geq 0 & \text{for } j \in N_1 \\
& x_j \leq 0 & \text{for } j \in N_2 \\
& x_j \text{ free} & \text{for } j \in N_3
\end{array}
$$

Obtain a lower bound:

$$
\begin{array}{lll}
\max & p^T \cdot b & \\
\text{s.t.} & p_i \geq 0 & \text{for } i \in M_1 \\
& p_i \leq 0 & \text{for } i \in M_2 \\
& p_i \text{ free} & \text{for } i \in M_3 \\
& A_j^T \cdot p \leq c_j & \text{for } j \in N_1 \\
& A_j^T \cdot p \geq c_j & \text{for } j \in N_2 \\
& A_j^T \cdot p = c_j & \text{for } j \in N_3
\end{array}
$$

Primal and Dual Linear Program

Consider the general linear program:

$$
\begin{array}{lll}
\min & c^T \cdot x & \\
\text{s.t.} & {a_i}^T \cdot x \geq b_i & \text{for } i \in M_1 \\
 & {a_i}^T \cdot x \leq b_i & \text{for } i \in M_2 \\
 & {a_i}^T \cdot x = b_i & \text{for } i \in M_3 \\
 & x_j \geq 0 & \text{for } j \in N_1 \\
 & x_j \leq 0 & \text{for } j \in N_2 \\
 & x_j \text{ free} & \text{for } j \in N_3
\end{array}
$$

Obtain a lower bound:

$$
\begin{array}{lll}
\max & p^T \cdot b & \\
\text{s.t.} & p_i \geq 0 & \text{for } i \in M_1 \\
 & p_i \leq 0 & \text{for } i \in M_2 \\
 & p_i \text{ free} & \text{for } i \in M_3 \\
 & A_j^T \cdot p \leq c_j & \text{for } j \in N_1 \\
 & A_j^T \cdot p \geq c_j & \text{for } j \in N_2 \\
 & A_j^T \cdot p = c_j & \text{for } j \in N_3
\end{array}
$$

The linear program on the right hand side is the dual linear program of the primal linear program on the left hand side.

This is a definition!

Primal and Dual Variables and Constraints

	primal LP (minimize)	dual LP (maximize)	
constraints	$\geq b_i$	≥ 0	variables
	$\leq b_i$	≤ 0	
	$= b_i$	free	
variables	≥ 0	$\leq c_i$	constraints
	≤ 0	$\geq c_i$	
	free	$= c_i$	

Examples

primal LP	dual LP
$\begin{array}{lrl} \min & c^T \cdot x & \\ \text{s.t.} & A \cdot x & \geq b \\ & x & \text{free} \end{array}$	$\begin{array}{lrl} \max & p^T \cdot b & \\ \text{s.t.} & A^T \cdot p & = c \\ & p & \geq 0 \end{array}$
$\begin{array}{lrl} \min & c^T \cdot x & \\ \text{s.t.} & A \cdot x & = b \\ & x & \geq 0 \end{array}$	$\begin{array}{lrl} \max & p^T \cdot b & \\ \text{s.t.} & A^T \cdot p & \leq c \\ & p & \text{free} \end{array}$

Basic Properties of the Dual Linear Program

Theorem 4.1.

The dual of the dual LP is the primal LP.

Proof:

Primal in general form:

$$
\begin{array}{lll}
\min & c^T \cdot x & \\
\text{s.t.} & {a_i}^T \cdot x \geq b_i & \text{for } i \in M_1 \\
& {a_i}^T \cdot x \leq b_i & \text{for } i \in M_2 \\
& {a_i}^T \cdot x = b_i & \text{for } i \in M_3 \\
& x_j \geq 0 & \text{for } j \in N_1 \\
& x_j \leq 0 & \text{for } j \in N_2 \\
& x_j \text{ free} & \text{for } j \in N_3
\end{array}
$$

Dual:

$$
\begin{array}{lll}
\max & p^T b & \\
\text{s.t.} & p_i \geq 0 & i \in M_1 \\
& p_i \leq 0 & i \in M_2 \\
& p_i \text{ free} & i \in M_3 \\
& A_j^T p \leq c_j & j \in N_1 \\
& A_j^T p \geq c_j & j \in N_2 \\
& A_j^T p = c_j & j \in N_3
\end{array}
$$

Basic Properties of the Dual Linear Program

Theorem 4.1.

The dual of the dual LP is the primal LP.

Proof:

Primal in general form:

$$
\begin{array}{lll}
\min & c^T \cdot x & \\
\text{s.t.} & a_i{}^T \cdot x \geq b_i & \text{for } i \in M_1 \\
& a_i{}^T \cdot x \leq b_i & \text{for } i \in M_2 \\
& a_i{}^T \cdot x = b_i & \text{for } i \in M_3 \\
& x_j \geq 0 & \text{for } j \in N_1 \\
& x_j \leq 0 & \text{for } j \in N_2 \\
& x_j \text{ free} & \text{for } j \in N_3
\end{array}
$$

Dual:

$$
\begin{array}{lll}
\max & p^T \cdot b & \\
\text{s.t.} & p_i \geq 0 & \text{for } i \in M_1 \\
& p_i \leq 0 & \text{for } i \in M_2 \\
& p_i \text{ free} & \text{for } i \in M_3 \\
& A_j^T \cdot p \leq c_j & \text{for } j \in N_1 \\
& A_j^T \cdot p \geq c_j & \text{for } j \in N_2 \\
& A_j^T \cdot p = c_j & \text{for } j \in N_3
\end{array}
$$

Basic Properties of the Dual Linear Program

Proof (cont.):

Dual:

$$
\begin{array}{llll}
\max & p^T \cdot b & & \\
\text{s.t.} & p_i \geq 0 & & \text{for } i \in M_1 \\
& p_i \leq 0 & & \text{for } i \in M_2 \\
& p_i \text{ free} & & \text{for } i \in M_3 \\
& A_j^T \cdot p \leq c_j & & \text{for } j \in N_1 \\
& A_j^T \cdot p \geq c_j & & \text{for } j \in N_2 \\
& A_j^T \cdot p = c_j & & \text{for } j \in N_3
\end{array}
$$

Dual (in primal form):

$\min \ -p^T b$

s.t.

||

Basic Properties of the Dual Linear Program

Proof (cont.):

Dual:

$$
\begin{array}{llll}
\max & p^T \cdot b & & \cdot(-1) \\
\text{s.t.} & p_i \geq 0 & \text{for } i \in M_1 & \\
& p_i \leq 0 & \text{for } i \in M_2 & \\
& p_i \text{ free} & \text{for } i \in M_3 & \\
& A_j^T \cdot p \leq c_j & \text{for } j \in N_1 & \cdot(-1) \\
& A_j^T \cdot p \geq c_j & \text{for } j \in N_2 & \cdot(-1) \\
& A_j^T \cdot p = c_j & \text{for } j \in N_3 &
\end{array}
$$

Dual (in primal form):

$$
\begin{array}{lll}
\min & -p^T \cdot b & \\
\text{s.t.} & -A_j^T \cdot p \geq -c_j & \text{for } j \in N_1 \\
& -A_j^T \cdot p \leq -c_j & \text{for } j \in N_2 \\
& -A_j^T \cdot p = -c_j & \text{for } j \in N_3 \\
& p_i \geq 0 & \text{for } i \in M_1 \\
& p_i \leq 0 & \text{for } i \in M_2 \\
& p_i \text{ free} & \text{for } i \in M_3
\end{array}
$$

Basic Properties of the Dual Linear Program

Proof (cont.):

Dual (in primal form):

$$
\begin{array}{lll}
\min & -p^T \cdot b & \\
\text{s.t.} & -A_j^T \cdot p \geq -c_j & \text{for } j \in N_1 \\
 & -A_j^T \cdot p \leq -c_j & \text{for } j \in N_2 \\
 & -A_j^T \cdot p = -c_j & \text{for } j \in N_3 \\
 & p_i \geq 0 & \text{for } i \in M_1 \\
 & p_i \leq 0 & \text{for } i \in M_2 \\
 & p_i \text{ free} & \text{for } i \in M_3
\end{array}
$$

Dual of Dual:

$$
\begin{array}{lll}
\max & -c^T x & \\
\text{s.t.} & x_j \geq 0 & j \in N_1 \\
 & x_j \leq 0 & j \in N_2 \\
 & x_j \text{ free} & j \in N_3 \\
 & -a_i^T x \leq -b_i & i \in M_1 \\
 & -a_i^T x \geq -b_i & i \in M_2 \\
 & -a_i^T x = -b_i & i \in M_3
\end{array}
$$

Equavalence of the Dual LP

Theorem 4.2.

Let Π_1 and Π_2 be two LPs where Π_2 has been obtained from Π_1 by (several) transformations of the following type:

- i replace a free variable by the difference of two non-negative variables;
- ii introduce a slack variable in order to replace an inequality constraint by an equation;
- iii if some row of a feasible equality system is a linear combination of the other rows, eliminate this row.

Then the dual of Π_1 is equivalent to the dual of Π_2.

Weak Duality Theorem

Theorem 4.3.

If x is a feasible solution to the primal LP (minimization problem) and p a feasible solution to the dual LP (maximization problem), then

$$c^T \cdot x \geq p^T \cdot b \quad .$$

Weak Duality Theorem

Theorem 4.3.

If x is a feasible solution to the primal LP (minimization problem) and p a feasible solution to the dual LP (maximization problem), then

$$c^T \cdot x \geq p^T \cdot b \ .$$

Corollary 4.4.

Consider a primal-dual pair of linear programs as above.

a. If the primal LP is unbounded (i. e., optimal cost $= -\infty$), then the dual LP is infeasible.
b. If the dual LP is unbounded (i. e., optimal cost $= \infty$), then the primal LP is infeasible.
c. If x and p are feasible solutions to the primal and dual LP, resp., and if $c^T \cdot x = p^T \cdot b$, then x and p are optimal solutions.

Proof of Thm 4.3:

By Thm 4.2 we may assume that primal (P) and dual (D) are of the form:

(P) $\min c^T x = \sum_{j=1}^{n} c_j x_j$

s.t. $Ax \geq b \Leftrightarrow \sum_{j=1}^{n} a_{ij} x_j \geq b_i, \forall i$

$x \geq 0$ $\quad x_i \geq 0 \quad \forall i$

(D) $\max p^T b = \sum_{i=1}^{m} b_i p_i$

s.t. $A^T p \leq c \Leftrightarrow \sum_{i=1}^{m} a_{ij} p_i \leq c_j \quad \forall j$

$p \geq 0$ $\quad p_i \geq 0 \quad \forall j$