Weak Duality Theorem

Theorem 4.3.

If x is a feasible solution to the primal LP (minimization problem) and p a feasible solution to the dual LP (maximization problem), then

$$c^T \cdot x \geq p^T \cdot b \ .$$

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Corollary 4.4.

Consider a primal-dual pair of linear programs as above.

a. If the primal LP is unbounded (i. e., optimal cost $= -\infty$), then the dual LP is infeasible.
b. If the dual LP is unbounded (i. e., optimal cost $= \infty$), then the primal LP is infeasible.
c. If x and p are feasible solutions to the primal and dual LP, resp., and if $c^T \cdot x = p^T \cdot b$, then x and p are optimal solutions.

Proof of Thm 4.3:

By Thm 4.2 we may assume that primal (P) and dual (D) are of the form:

$$(P)\ \min c^T x = \sum_{j=1}^{n} c_j x_j \qquad \text{s.t. } Ax \geq b \iff \sum_{j=1}^{n} a_{ij} x_j \geq b_i,\ \forall i \qquad x \geq 0 \quad x_j \geq 0\ \forall j$$

$$(D)\ \max p^T b = \sum_{i=1}^{m} b_i p_i \qquad \text{s.t. } A^T p \leq c \iff \sum_{i=1}^{m} a_{ij} p_i \leq c_j\ \forall j \qquad p \geq 0 \quad p_i \geq 0\ \forall i$$

Let x, p be feasible solutions for (P) and (D), respectively.

$$c^T x \geq (A^T p)^T x = (p^T A) x = p^T (Ax) \geq p^T b$$

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij} p_i x_j = \sum_{i=1}^{m} p_i \sum_{j=1}^{n} a_{ij} x_j \geq \sum_{i=1}^{m} p_i b_i$$

Strong Duality Theorem

identical to: $A^T p \le c$

Theorem 4.5.

If an LP has an optimal solution, so does its dual and the optimal costs are equal.

Proof: Assume that (P) has an optimal solution

By Thm 4.2, we may assume

(P) $\min c^T x$
s.t. $Ax = b$
$x \ge 0$

(D) $\max p^T b$
s.t. $p^T A \le c^T$
p free

Optimum is attained in some bfs with bases B

Let y be the optimum,

Simplex tableau corresponding to B:

$$\begin{array}{c|c} -c^T y & c^T - c_B^T B^{-1} A \\ \hline B^{-1} b & B^{-1} A \end{array}$$

reduced costs

y optimal $\Rightarrow$ reduced costs ≥ 0

$$c^T - \underbrace{c_B^T B^{-1}}_{=: q^T} A \geq 0$$

$\Rightarrow q^T A \leq c^T$

$\Rightarrow q$ is feasible in (D)

Then

$$q^T b = \left(c_B^T B^{-1}\right) b = c_B^T \left(B^{-1} b\right) = c_B^T y_B = c^T y$$

$\Rightarrow q^T b = c^T y$

$\Rightarrow q$ is optimal in (D) by weak duality. $\square$

Different Possibilities for Primal and Dual LP

primal \ dual	finite optimum	unbounded	infeasible
finite optimum	possible	impossible	impossible
unbounded	impossible	impossible	possible
infeasible	impossible	possible	possible

strong duality

weak duality

Different Possibilities for Primal and Dual LP

primal \ dual	finite optimum	unbounded	infeasible
finite optimum	possible	impossible	impossible
unbounded	impossible	impossible	possible
infeasible	impossible	possible	possible

Example of infeasible primal and dual LP:

$$\begin{aligned} \min \quad & x_1 + 2\,x_2 \\ \text{s.t.} \quad & x_1 + x_2 = 1 \\ & 2\,x_1 + 2\,x_2 = 3 \end{aligned}$$

$$\begin{aligned} \max \quad & p_1 + 3\,p_2 \\ \text{s.t.} \quad & p_1 + 2\,p_2 = 1 \\ & p_1 + 2\,p_2 = 2 \end{aligned}$$