

Weak Duality Theorem

Theorem 4.3.

If x is a feasible solution to the primal LP (minimization problem) and p a feasible solution to the dual LP (maximization problem), then

$$c^T \cdot x \geq p^T \cdot b .$$

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Corollary 4.4.

Consider a primal-dual pair of linear programs as above.

- a** If the primal LP is unbounded (i. e., optimal cost = $-\infty$), then the dual LP is infeasible.
- b** If the dual LP is unbounded (i. e., optimal cost = ∞), then the primal LP is infeasible.
- c** If x and p are feasible solutions to the primal and dual LP, resp., and if $c^T \cdot x = p^T \cdot b$, then x and p are optimal solutions.

Proof of Thm 4.3:

By Thm 4.2 we may assume that primal (P) and dual (D) are of the form:

$$(P) \min c^T x = \sum_{j=1}^n c_j x_j$$

s.t. $Ax \geq b$
 $x \geq 0$

$\Leftrightarrow \sum_{j=1}^n a_{ij} x_j \geq b_i, \forall i$
 $x_j \geq 0 \forall j$

$$(D) \max p^T b = \sum_{i=1}^m b_i p_i$$

s.t. $A^T p \leq c$
 $p \geq 0$

$\Leftrightarrow \sum_{i=1}^m a_{ij} p_i \leq c_j$
 $p_i \geq 0 \forall i$

Let x, p be feasible solutions for (P) and (D), respectively.

$$c^T x \geq (A^T p)^T x = (p^T A) x = p^T (Ax) \geq p^T b$$

$$\sum_{j=1}^n c_j x_j \geq \sum_{i=1}^m \sum_{j=1}^n a_{ij} p_i x_j = \sum_{i=1}^m p_i \sum_{j=1}^n a_{ij} x_j \geq \sum_{i=1}^m p_i b_i$$

Strong Duality Theorem

Theorem 4.5.

If an LP has an optimal solution, so does its dual and the optimal costs are equal.

Proof: Assume that (P) has an optimal solution

By Thm 4.2, we may assume

$$(P) \min c^T x \\ \text{s.t. } Ax = b \\ x \geq 0$$

$$(D) \max p^T b \\ \text{s.t. } p^T A \leq c^T \\ p \text{ free}$$

Optimum is attained in some bfs with bases B

Let y be the optimum,

Simplex tableau corresponding to B :

$-c^T y$	$c^T - c_B^T B^{-1} A$
$B^{-1} b$	$B^{-1} A$

reduced costs

identical
+ 0: $A^T p \leq c$

y optimal \Rightarrow reduced costs $\geq \sigma$

$$c^T - \underbrace{c_B^T B^{-1} A}_{=: q^T} \geq \sigma$$

$$\Rightarrow q^T A \leq c^T$$

$\Rightarrow q$ is feasible in (D)

Then

$$q^T b = (c_B^T B^{-1}) b = c_B^T (B^{-1} b) = c_B^T y_B = c^T y$$

$$\Rightarrow q^T b = c^T y$$

$\Rightarrow q$ is optimal in (D) by weak duality.

□

Different Possibilities for Primal and Dual LP

primal \ dual	finite optimum	unbounded	infeasible
finite optimum	possible	impossible	impossible
unbounded	impossible	impossible	possible
infeasible	impossible	possible	possible

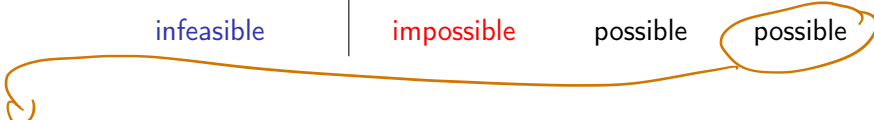
strong duality

weak duality



Different Possibilities for Primal and Dual LP

primal \ dual	finite optimum	unbounded	infeasible
finite optimum	possible	impossible	impossible
unbounded	impossible	impossible	possible
infeasible	impossible	possible	possible



Example of infeasible primal and dual LP:

$$\begin{array}{ll} \min & x_1 + 2x_2 \\ \text{s.t.} & x_1 + x_2 = 1 \\ & 2x_1 + 2x_2 = 3 \end{array}$$

$$\begin{array}{ll} \max & p_1 + 3p_2 \\ \text{s.t.} & p_1 + 2p_2 = 1 \\ & p_1 + 2p_2 = 2 \end{array}$$