Complementary Slackness

Consider the following pair of primal and dual LPs:

$$\begin{aligned} \min \quad & c^T \cdot x \\ \text{s.t.} \quad & A \cdot x \geq b \end{aligned} \qquad\qquad \begin{aligned} \max \quad & p^T \cdot b \\ \text{s.t.} \quad & p^T \cdot A = c^T \\ & p \geq 0 \end{aligned}$$

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$$c^T \cdot x = p^T \cdot b \iff \text{for all } i\text{: } p_i = 0 \text{ if } {a_i}^T \cdot x > b_i.$$

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$$c^T \cdot x = p^T \cdot b \quad \Longleftrightarrow \quad \text{for all } i\text{: } p_i = 0 \text{ if } {a_i}^T \cdot x > b_i.$$

Theorem 4.6. (Complementary Slackness Theorem)

Consider an arbitrary pair of primal and dual LPs. Let x and p be feasible solutions to the primal and dual LP, respectively. Then x and p are both optimal if and only if

$$u_i := p_i({a_i}^T \cdot x - b_i) = 0 \quad \text{for all } i, \tag{1}$$

$$v_j := (c_j - p^T \cdot A_j)\, x_j = 0 \quad \text{for all } j. \tag{2}$$

Proof of Thm 4.6:

- $u_i \geq 0$ since $a_i^T x - b_i = 0 \Rightarrow u_i = 0$

  $a_i^T x - b_i > 0 \Rightarrow p_i \geq 0 \Rightarrow u_i \geq 0$

  $a_i^T x - b_i < 0 \Rightarrow p_i \leq 0 \Rightarrow u_i \geq 0$

- $v_j \geq 0$ similarly.

- Thus $U := \sum_i u_i \geq 0$ $\qquad U = 0 \Leftrightarrow (1)$ holds

  $V := \sum_j v_j \geq 0$ $\qquad V = 0 \Leftrightarrow (2)$ holds

- $$U + V = \sum_i p_i (a_i^T x - b_i) + \sum_j (c_j - p^T A_j) x_j$$

  $$= -\sum_i p_i b_i + \sum_j c_j x_j + \sum_i p_i a_i^T x - \sum_j p^T A_j x_j$$

  $$= c^T x - p^T b + \underbrace{\sum_i p_i \sum_j a_{ij} x_j - \sum_j \sum_i a_{ij} p_i x_j}_{= 0}$$

- Suppose (1) and (2) hold.

$\Rightarrow u+v=0 \Rightarrow c^T x = p^T b$

$\Rightarrow$ by weak duality $\Rightarrow x, p$ optimal

- Suppose x and p are optimal

$\Rightarrow$ by strong duality $\Rightarrow c^T x = p^T b$

$\Rightarrow u+v=0$

$\Rightarrow$ (1) and (2) hold

Complementary Slackness Example

Consider the following LP in standard ~~for~~ form and its dual:

$$\begin{array}{llll} \min & 13x_1 + 10x_2 + 6x_3 & & \\ \text{s.t.} & 5x_1 + x_2 + 3x_3 & = & 8 \\ & 3x_1 + x_2 & = & 3 \\ & x_1, x_2, x_3 & \geq & 0 \end{array} \qquad \begin{array}{llll} \max & 8p_1 + 3p_2 & & \\ \text{s.t.} & 5p_1 + 3p_2 & \leq & 13 \\ & p_1 + p_2 & \leq & 10 \\ & 3p_1 & \leq & 6 \end{array}$$

$c^T x = 19$

Claim: $x^* = (1, 0, 1)$ is a non-degenerate optimal solution to the primal.

Verify this using complementary slackness!

- $p_i (a_i^T x^* - b_i) = 0$ is automatically satisfied since primal is in standard form
- $(c_j - p^T A_j) x_j^* = 0$ $\Rightarrow$ satisfied for $j=2$ as $x_2^* = 0$

$$\left.\begin{array}{ll} j=1: & 5p_1 + 3p_2 = 13 \\ j=3: & 3p_1 = 6 \end{array}\right\} \Rightarrow p_1 = 2, \; p_2 = 1$$

$\hookrightarrow$ dual feasible as $p_1 + p_2 \leq 10$

$8p_1 + 3p_2 = 19$ $\Rightarrow$ optimal