Geometric View

Consider pair of primal and dual LPs with $A \in \mathbb{R}^{m \times n}$ and $\text{rank}(A) = n$:

$$\begin{aligned} \min \quad & c^T \cdot x \\ \text{s.t.} \quad & {a_i}^T \cdot x \geq b_i, \quad i = 1, \dots, m \end{aligned}$$

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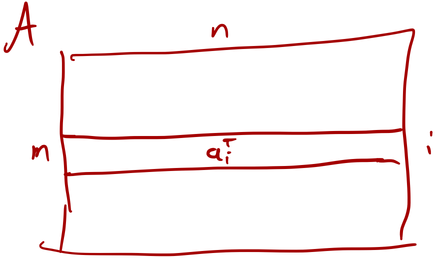

$$\sum_{i=1}^{m} \sum_{j=1}^{n} p_i a_{ij} = c_j \quad \forall j = 1..m$$

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- i ${a_i}^T \cdot x \geq b_i$ for all i (primal feasibility)
- ii $p_i = 0$ for all $i \notin I$ (complementary slackness)
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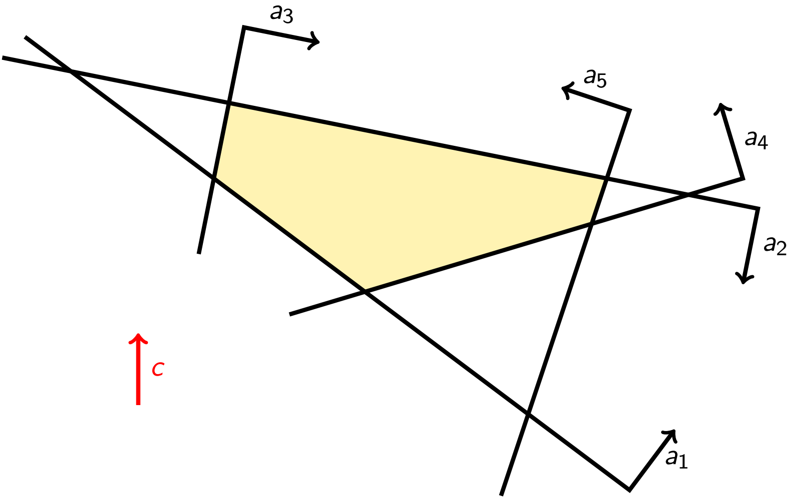
The a_i, $i \in I$, form basis for dual LP and p^I is corresponding basic solution.

Geometric View (cont.)



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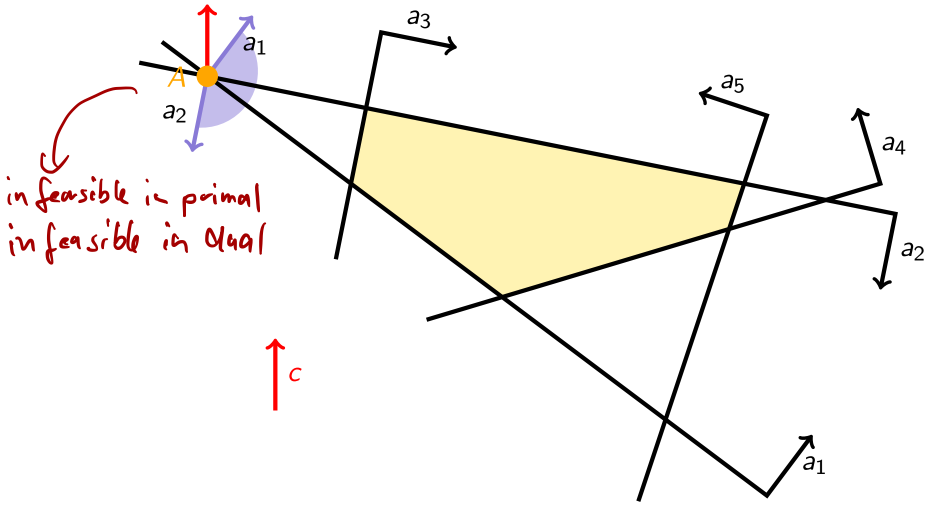

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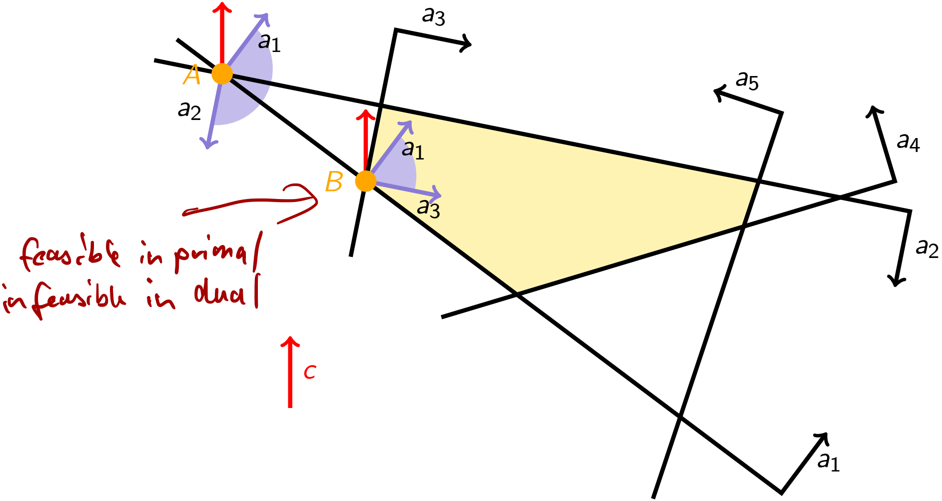

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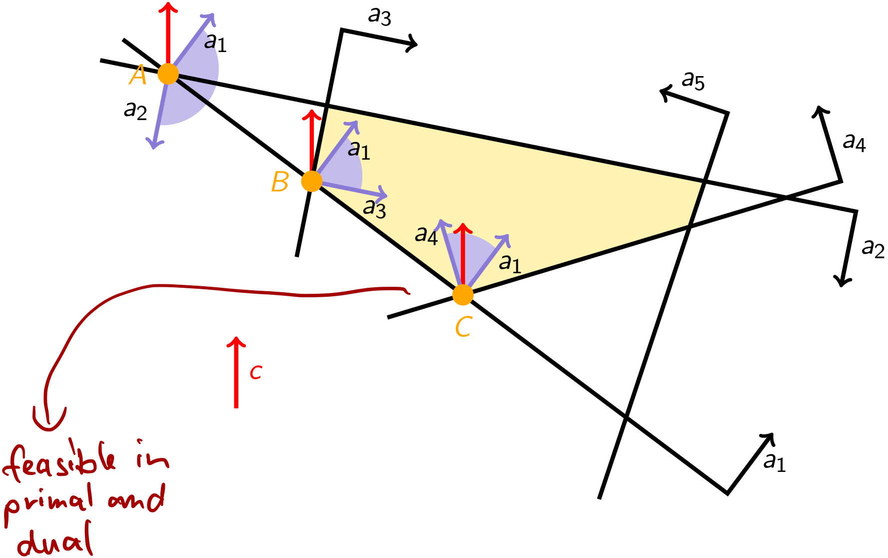

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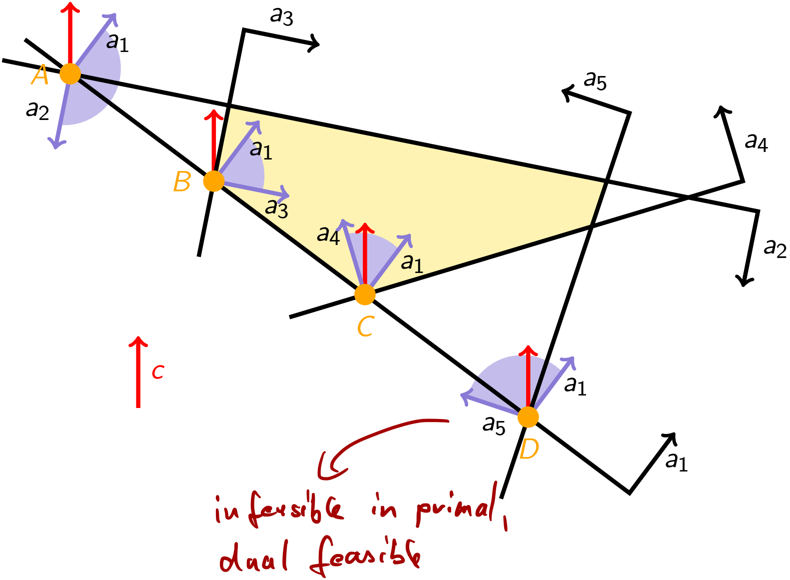

Dual Variables as Marginal Costs

Consider the primal dual pair:

$$\begin{aligned} \min \quad & c^T \cdot x \\ \text{s.t.} \quad & A \cdot x = b \\ & x \geq 0 \end{aligned} \qquad\qquad \begin{aligned} \max \quad & p^T \cdot b \\ \text{s.t.} \quad & p^T \cdot A \leq c^T \end{aligned}$$

$\Longleftrightarrow A^T p \leq c$

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Replace b by $b + d$. For small d, the basis B remains feasible and optimal:

$$B^{-1} \cdot (b + d) = B^{-1} \cdot b + B^{-1} \cdot d \geq 0 \qquad \text{(feasibility)}$$

reduced costs

$$\bar{c}^T = c^T - {c_B}^T \cdot B^{-1} \cdot A \geq 0 \qquad \text{(optimality)}$$

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Optimal cost of perturbed problem is

$$c_B{}^T \cdot B^{-1} \cdot (b + d) = c_B{}^T \cdot x_B^* + \underbrace{(c_B{}^T \cdot B^{-1})}_{= p^T} \cdot d$$

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Thus, p_i is the marginal cost per unit increase of b_i.

Dual Variables as Shadow Prices

Diet problem:

- $a_{ij} :=$ amount of nutrient i in one unit of food j
- $b_i :=$ requirement of nutrient i in some ideal diet
- $c_j :=$ cost of one unit of food j on the food market

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- p_i is "fair" price per unit of nutrient i

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- food j used in ideal diet ($x_j^* > 0$) is consistently priced at the two markets (by complementary slackness)
- ideal diet has the same value on both markets (by strong duality)