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Consider LP in standard form with $A \in \mathbb{R}^{m \times n}$, $\operatorname{rank}(A) = m$, and dual LP:

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b. p is feasible if and only if $\bar{c} \geq 0$; $\bar{c}^T = c^T - c_B^T B^{-1} A = c^T - p^T A \geq 0$
c. reduced cost $\bar{c}_i = 0$ corresponds to active dual constraint;
d. p is degenerate if and only if $\bar{c}_i = 0$ for some non-basic variable x_i.

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dual feasible

improve (weakly) dual solution

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II If $v_i \geq 0$ for all $i \in \{1, \dots, n\}$, then the dual LP is unbounded and the primal LP is infeasible.

Dual Simplex Example

		x_1	x_2	x_3	x_4	x_5
	0	2	6	10	0	0
$x_4 =$	2	-2	4	1	1	0
$x_5 =$	-1	4	-2	-3	0	1

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  - Column 2 and 3 have negative entries in pivot row.

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- Find pivot column.
  - Column 2 and 3 have negative entries in pivot row.
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- Perform basis change:
  - x_5 leaves and x_2 enters basis.
  - Eliminate other entries in the pivot column.
  - Divide pivot row by pivot element.

Dual Simplex Example

		x_1	x_2	x_3	x_4	x_5
	-3	14	0	1	0	3
$x_4 =$	2	-2	4	1	1	0
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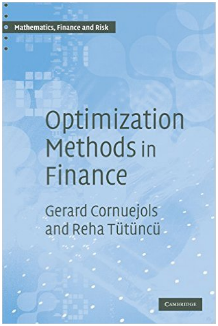
Remarks on the Dual Simplex Method

- Dual simplex method terminates if lexicographic pivoting rule is used:
  - Choose any row ℓ with $x_{B(\ell)} < 0$ to be the pivot row.
  - Among all columns j with $v_j < 0$ choose the one which is lexicographically minimal when divided by $|v_j|$.
- Dual simplex method is useful if, e. g., dual basic solution is readily available.
- Example: Resolve LP after right-hand-side b has changed.

COMP331/557

Chapter 5:
Optimisation in Finance: Cash-Flow

(Cornuejols & Tütüncü, Chapter 3)



Cash-Flow Management Problem

A company has the following net cash flow requirements (in 1000's of £):

Month	Jan	Feb	Mar	Apr	May	Jun
Net cash flow	-150	-100	200	-200	50	300

E.g.: In January we have to pay £150k and in March we get £200k.

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Initially we have no cash but the following possibilities to borrow/invest money:

i. a line of credit of up to £100k at an interest rate of 1% per month;
ii. in any one of the first three months, it can issue 90-day commercial paper bearing a total interest of 2% for the three-month period;
iii. excess funds can be invested at an interest rate of 0.3% per month.

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iii. excess funds can be invested at an interest rate of 0.3% per month.

Task: We want to maximise the companies wealth in June, while fulfilling all payments.

Cash-Flow Management Problem – Modelling as LP

Decision Variables

- v .. wealth in June
- x_i .. amount drawn from credit line in month i
- y_i .. amount of commercial paper issued in month i
- z_i .. excess funds in month i

LP formulation:

Cash-Flow Management Problem – Modelling as LP

Decision Variables

- v .. wealth in June
- x_i .. amount drawn from credit line in month i
- y_i .. amount of commercial paper issued in month i
- z_i .. excess funds in month i

LP formulation:

$$
\begin{array}{llrrrrrrrrrrrl}
\max & v & & & & & & & & & & & & \\
\text{s.t.} & x_1 & + & y_1 & & & & & - & z_1 & = & 150 & \\
 & x_2 & + & y_2 & - & 1.01x_1 & + & 1.003z_1 & - & z_2 & = & 100 & \\
 & x_3 & + & y_3 & - & 1.01x_2 & + & 1.003z_2 & - & z_3 & = & -200 & \\
 & x_4 & - & 1.02y_1 & - & 1.01x_3 & + & 1.003z_3 & - & z_4 & = & 200 & \\
 & x_5 & - & 1.02y_2 & - & 1.01x_4 & + & 1.003z_4 & - & z_5 & = & -50 & \\
 & & - & 1.02y_3 & - & 1.01x_5 & + & 1.003z_5 & - & v & = & -300 & \\
 & & & & & & & & & x_i & \leq & 100 & \forall i \\
 & & & & & & & x_i, & y_i, & z_i & \geq & 0 & \forall i
\end{array}
$$

Cash-Flow Management Problem – Modelling as LP

cashflow.lp

```
Maximize
  wealth:  v
Subject To
  Jan:  x1 + y1 - z1 = 150
  Feb:  x2 + y2 - 1.01 x1 + 1.003 z1 - z2 = 100
  Mar:  x3 + y3 - 1.01 x2 + 1.003 z2 - z3 = -200
  Apr:  x4 - 1.02 y1 - 1.01 x3 + 1.003 z3 - z4 = 200
  May:  x5 - 1.02 y2 - 1.01 x4 + 1.003 z4 - z5 = -50
  Jun:  - 1.02 y3 - 1.01 x5 + 1.003 z5 - v = -300
Bounds
  0 <= x1 <= 100
  0 <= x2 <= 100
  0 <= x3 <= 100
  0 <= x4 <= 100
  0 <= x5 <= 100
  -Inf <= v <= Inf
End
```