Cash-Flow Management Problem – Modelling as LP

cashflow.lp

```
Maximize
  wealth:  v
Subject To
  Jan:  x1 + y1 - z1 = 150
  Feb:  x2 + y2 - 1.01 x1 + 1.003 z1 - z2 = 100
  Mar:  x3 + y3 - 1.01 x2 + 1.003 z2 - z3 = -200
  Apr:  x4 - 1.02 y1 - 1.01 x3 + 1.003 z3 - z4 = 200
  May:  x5 - 1.02 y2 - 1.01 x4 + 1.003 z4 - z5 = -50
  Jun:  - 1.02 y3 - 1.01 x5 + 1.003 z5 - v = -300
Bounds
  0 <= x1 <= 100
  0 <= x2 <= 100
  0 <= x3 <= 100
  0 <= x4 <= 100
  0 <= x5 <= 100
  -Inf <= v <= Inf
End
```

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Gurobi Output

```
Solved in 5 iterations and 0.00 seconds
Optimal objective 9.249694915e+01
v 92.4969491525
x1 0.0
y1 150.0
z1 0.0
x2 0.0
y2 100.0
z2 0.0
x3 0.0
y3 151.944167498
z3 351.944167498
x4 0.0
z4 0.0
x5 52.0
z5 0.0

Obj:  92.4969491525
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Optimal Investment Strategy:

Jan: Issue commercial paper for £150k.

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- **Jun:** wealth $\approx$ £92k

COMP331/557

Chapter 6: Optimal Trees and Paths

(Cook, Cunningham, Pulleyblank & Schrijver, Chapter 2)

Trees and Forests

Definition 6.1.

- i An undirected graph having no circuit is called a forest.
- ii A connected forest is called a tree.

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v. G has no circuit. Adding an arbitrary edge to G creates a circuit.
vi. G contains a unique path between any pair of nodes.

Kruskal's Algorithm

Minimum Spanning Tree (MST) Problem

Given: connected graph $G = (V, E)$, cost function $c : E \to \mathbb{R}$.

Task: find spanning tree $T = (V, F)$ of G with minimum cost $\sum_{e \in F} c(e)$.

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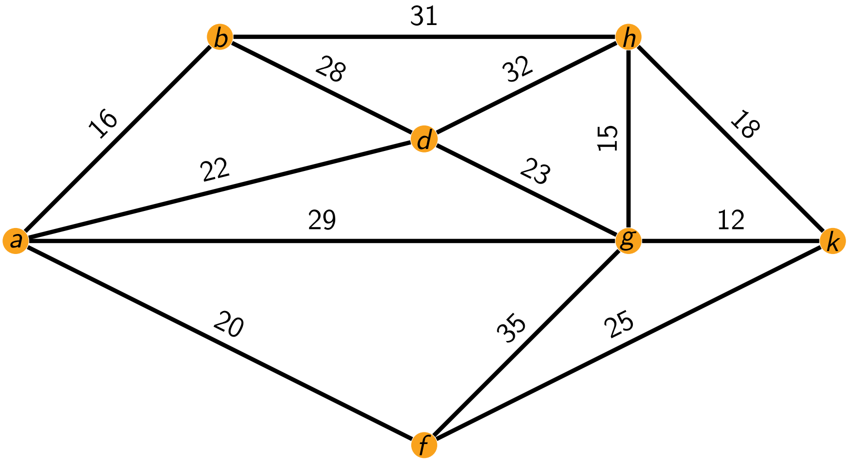
Kruskal's Algorithm for MST

1. Sort the edges in E such that $c(e_1) \leq c(e_2) \leq \cdots \leq c(e_m)$.
2. Set $T := (V, \emptyset)$. ← empty set
3. For $i := 1$ to m do:

    If adding e_i to T does not create a circuit, then add e_i to T.

Example for Kruskal's Algorithm



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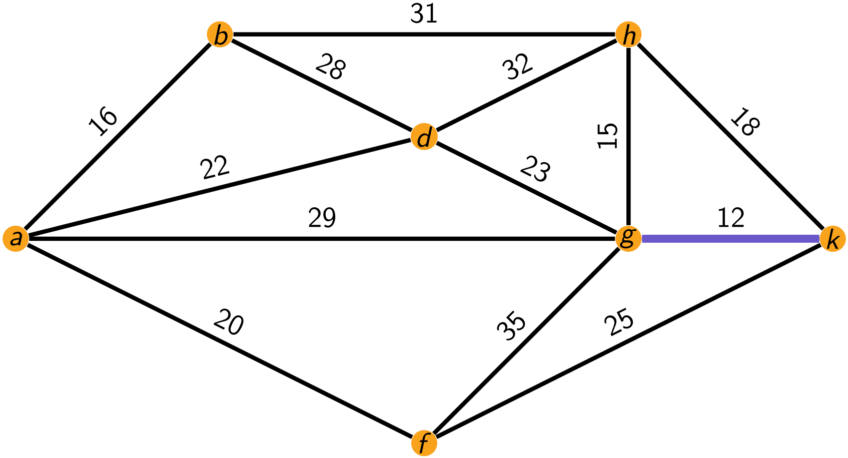

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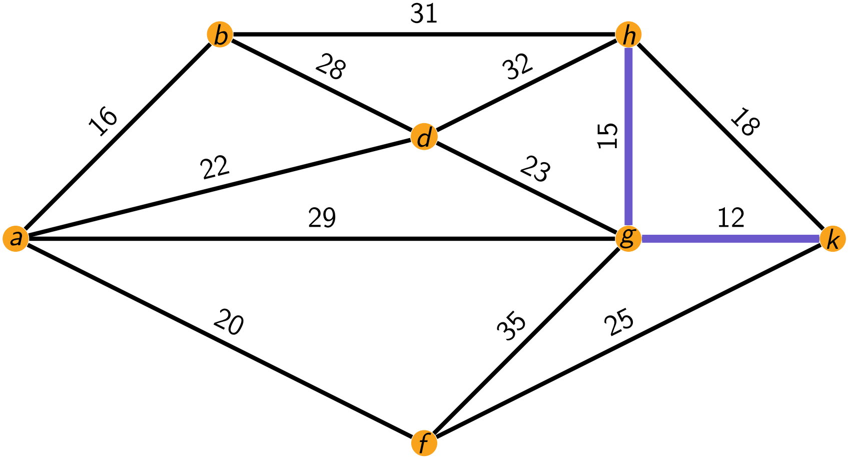

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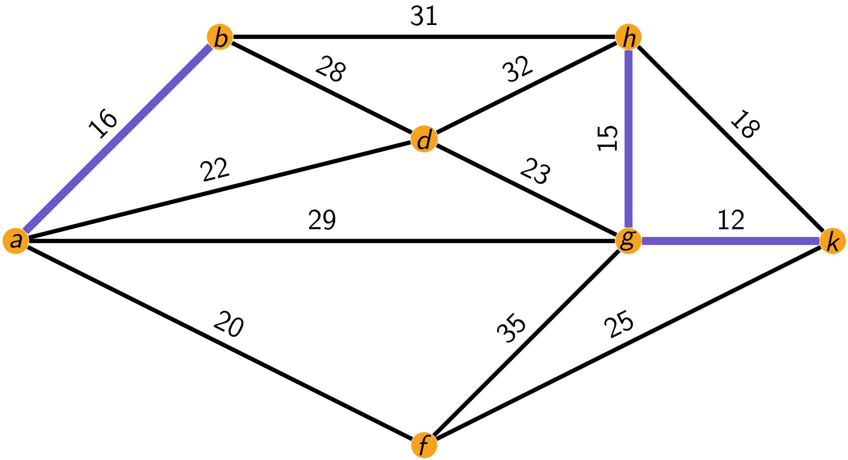

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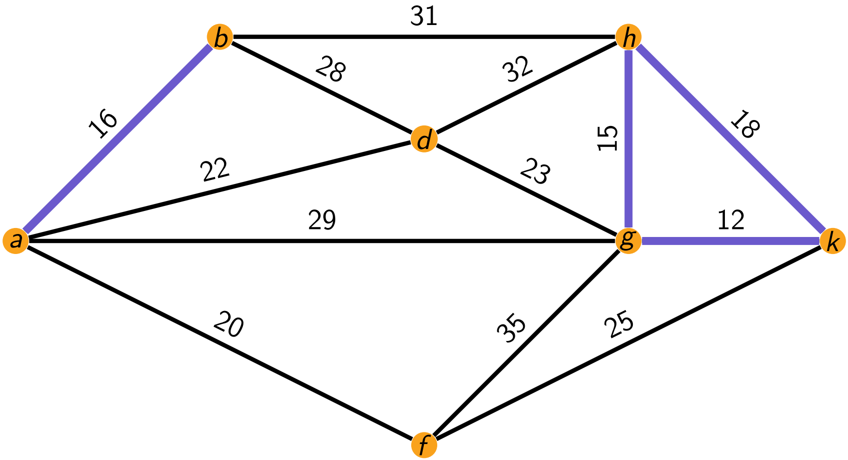

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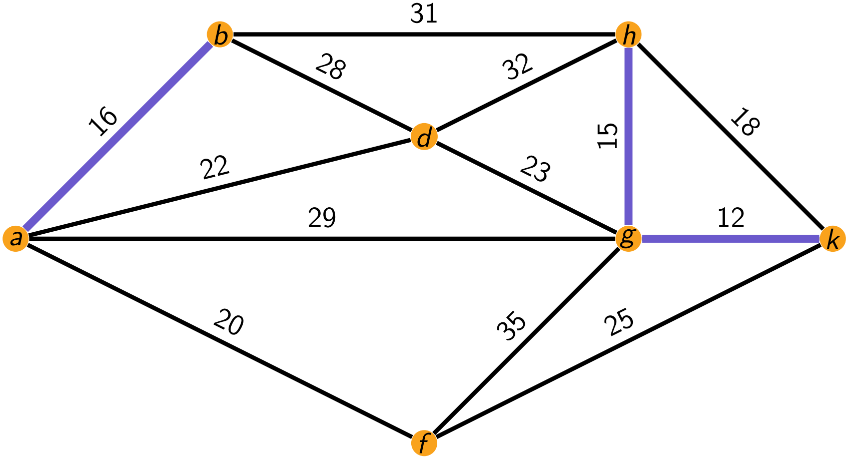

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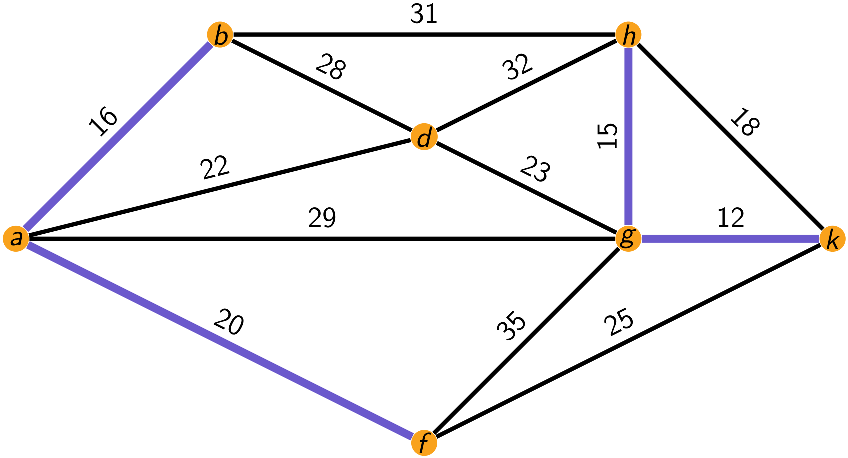

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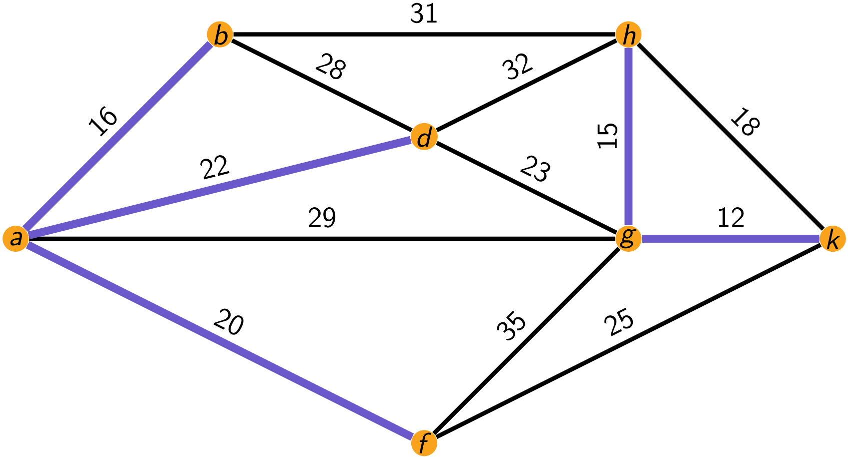

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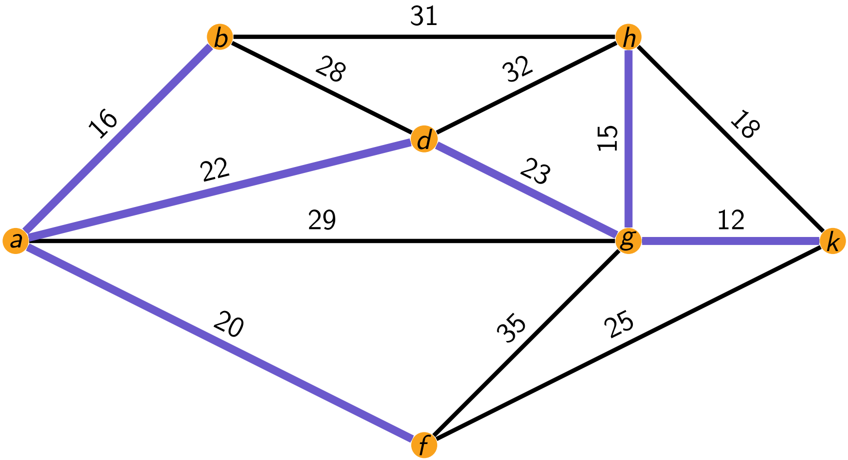

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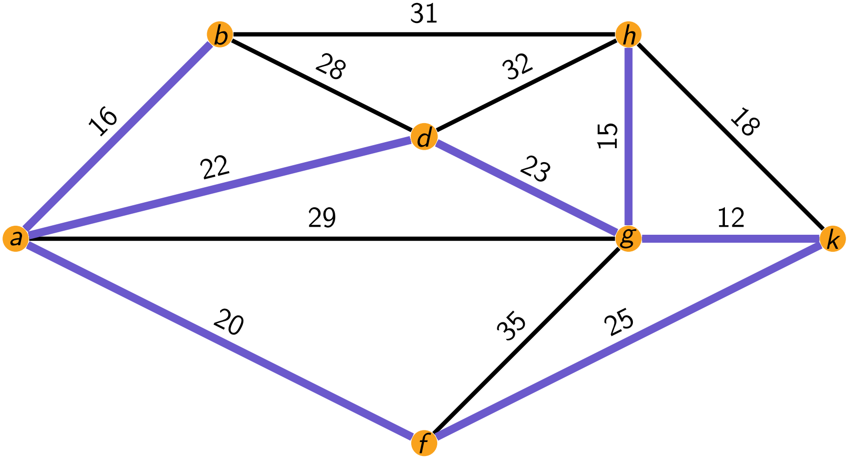

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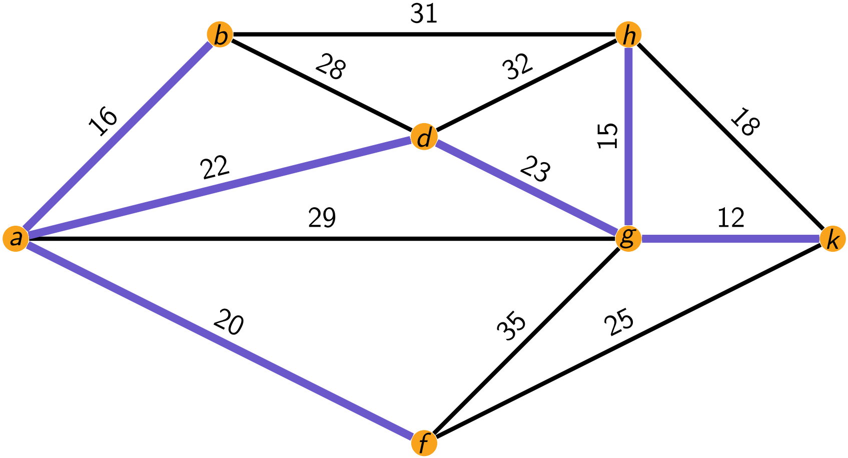

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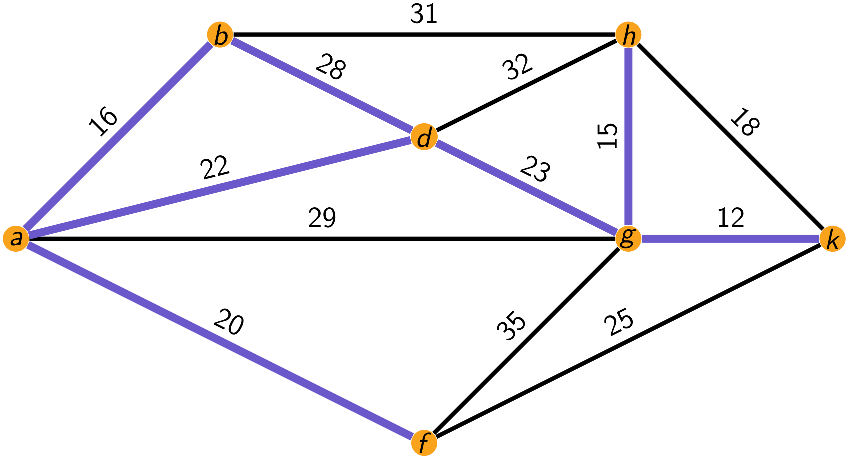

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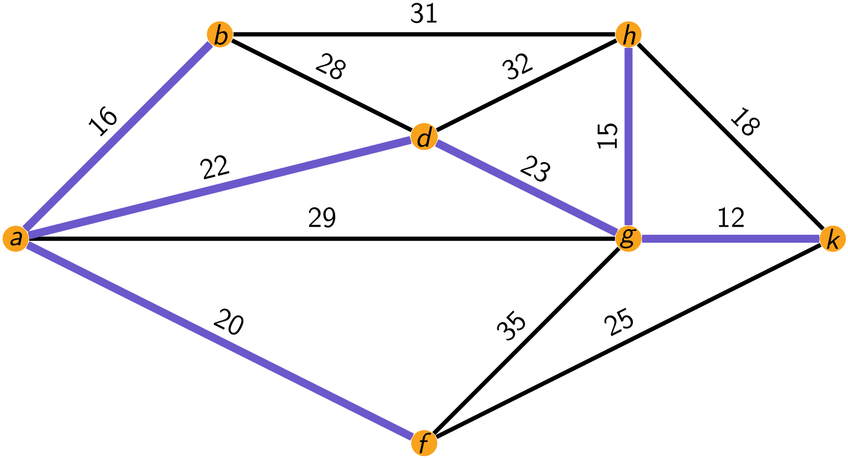

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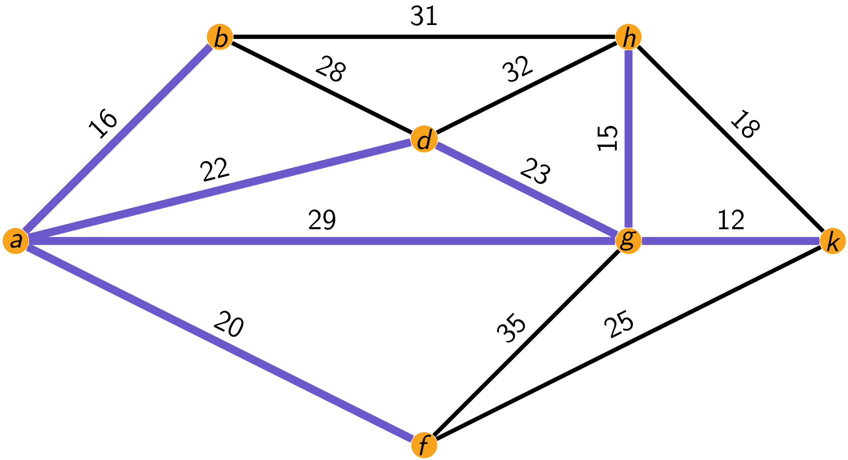

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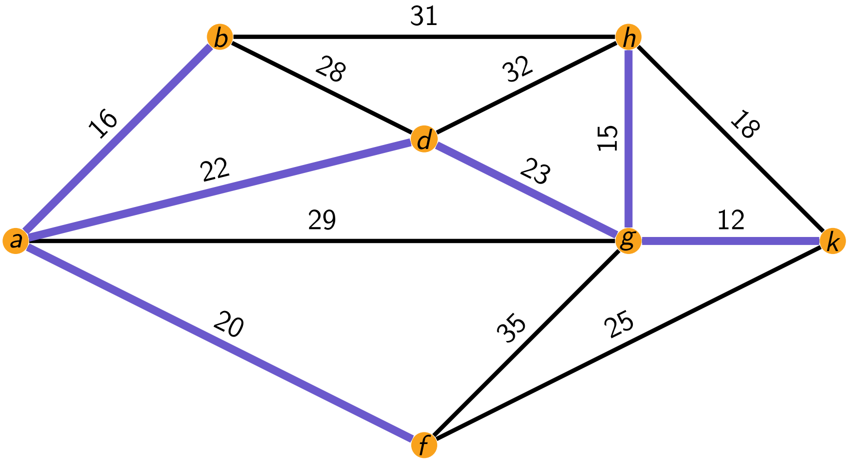

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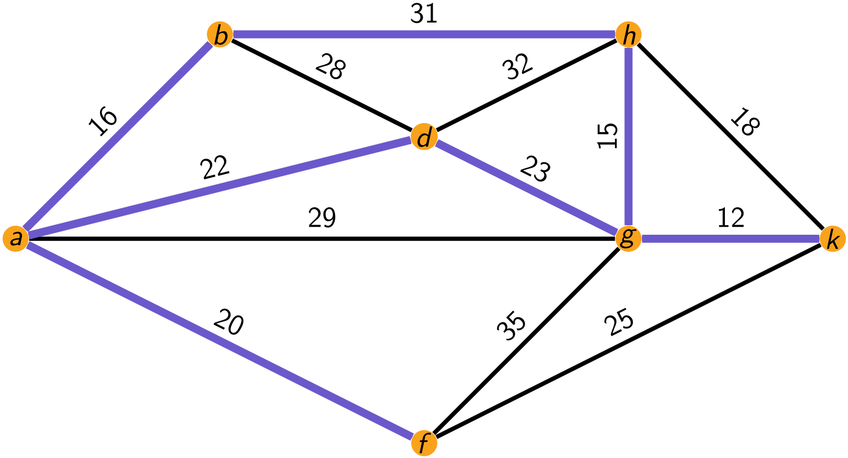

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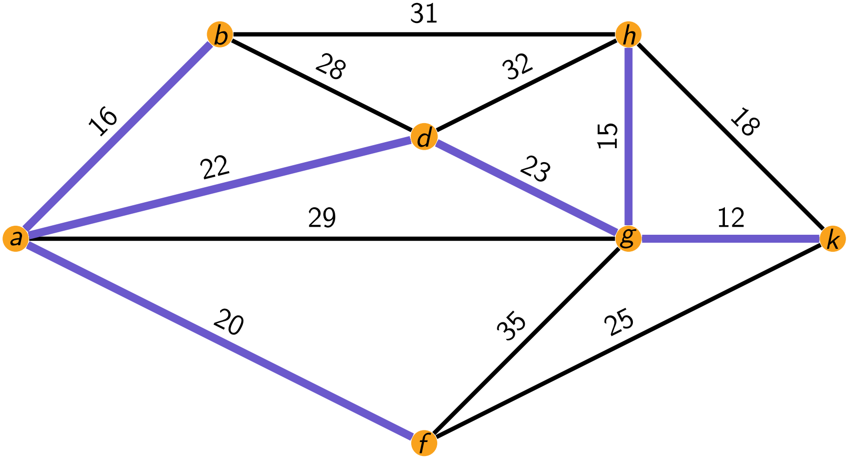

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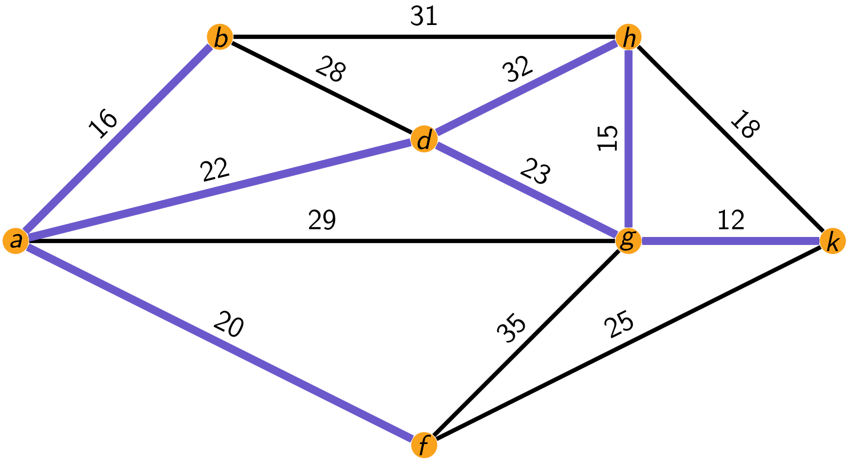

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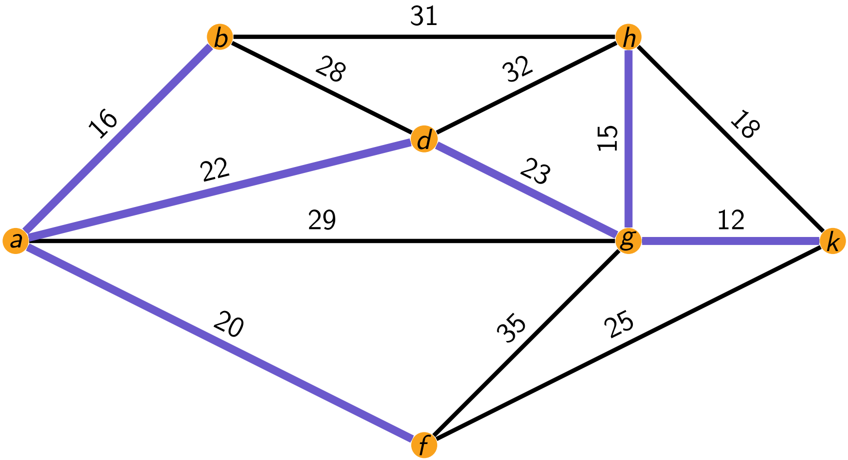

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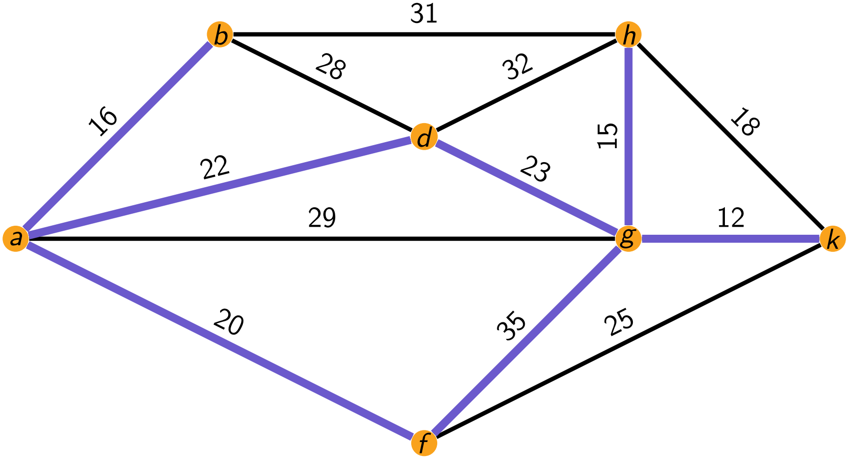

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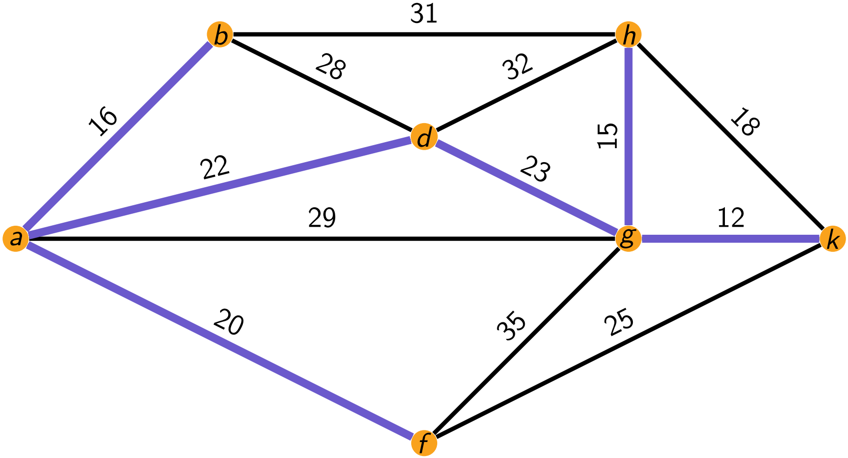
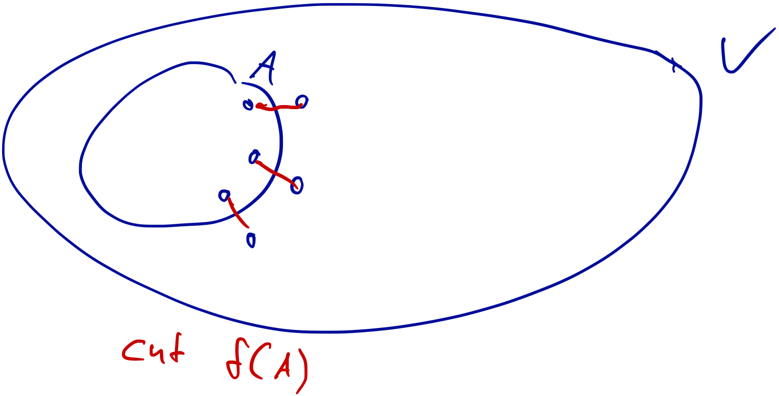


Prim's Algorithm

Notation: For a graph $G = (V, E)$ and $A \subseteq V$ let

$$\delta(A) := \{e = \{v, w\} \in E \mid v \in A \text{ and } w \in V \setminus A\} \ .$$

We call $\delta(A)$ the cut induced by A.



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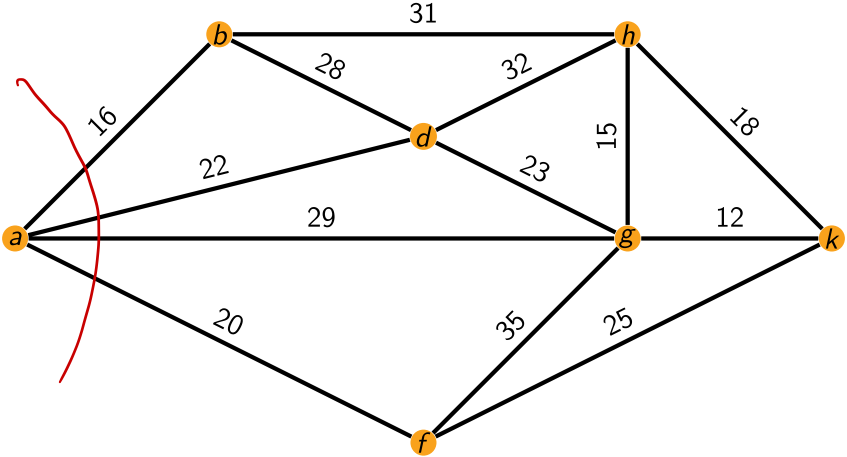

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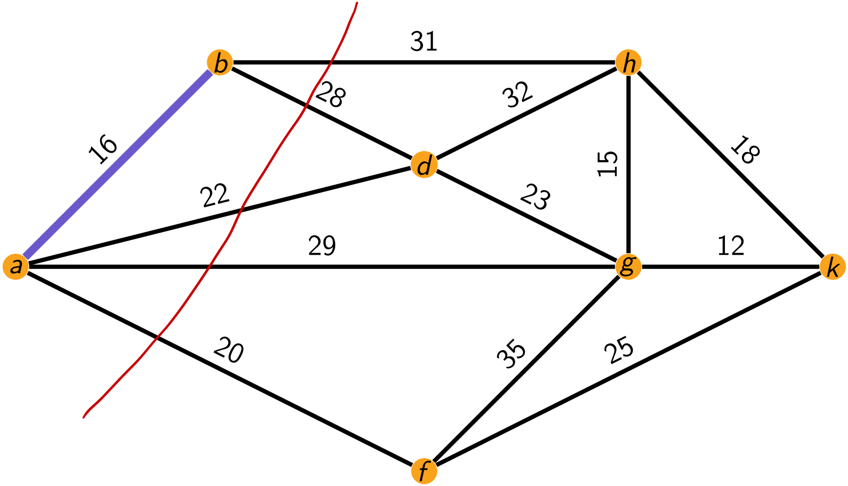

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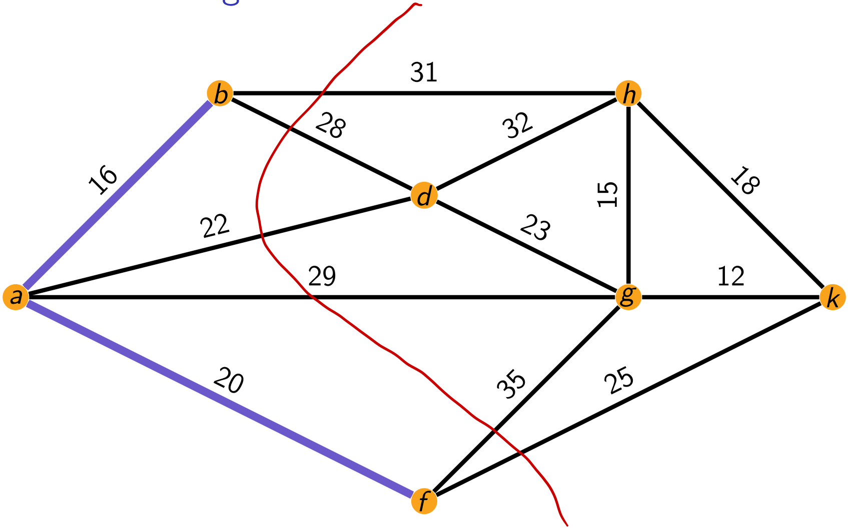

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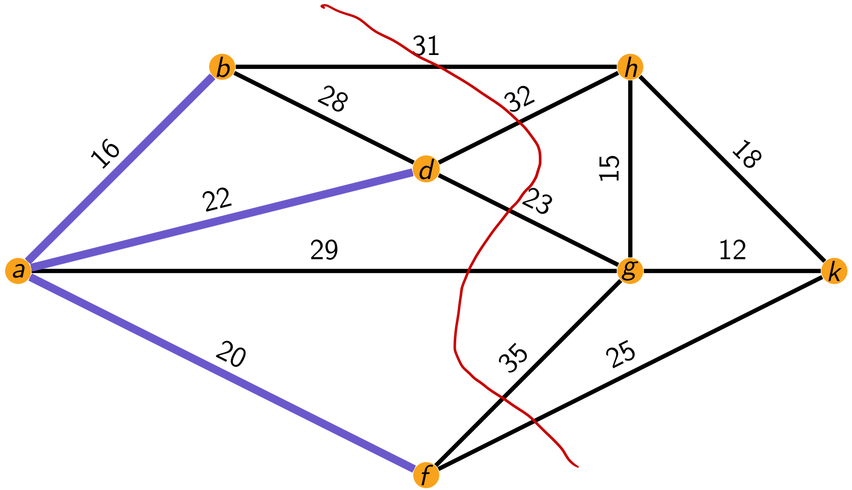

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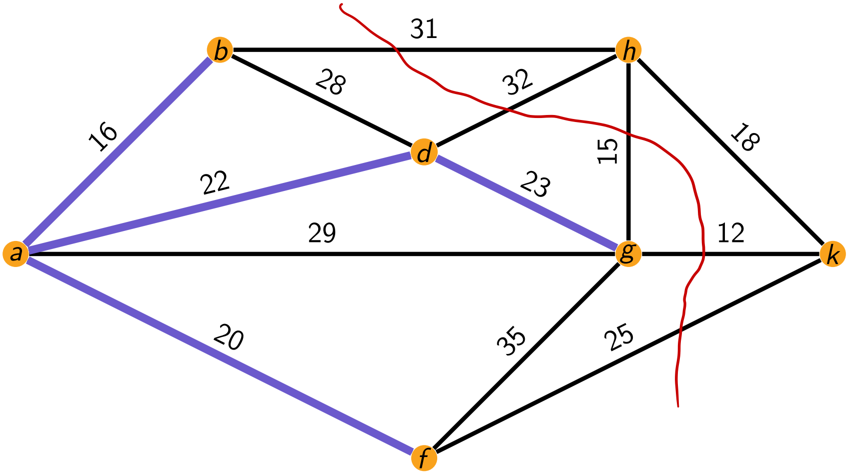

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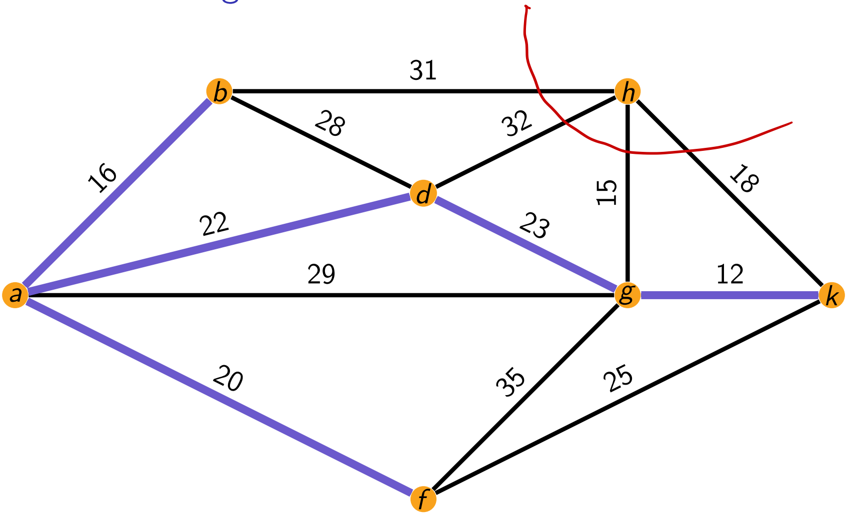

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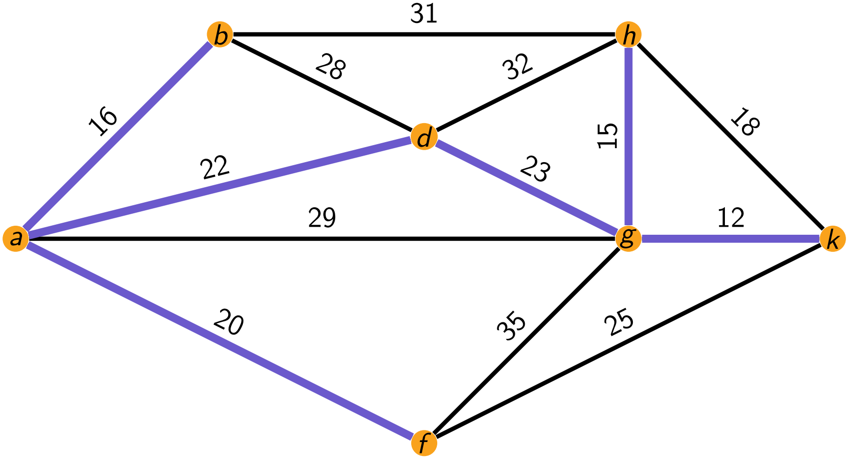
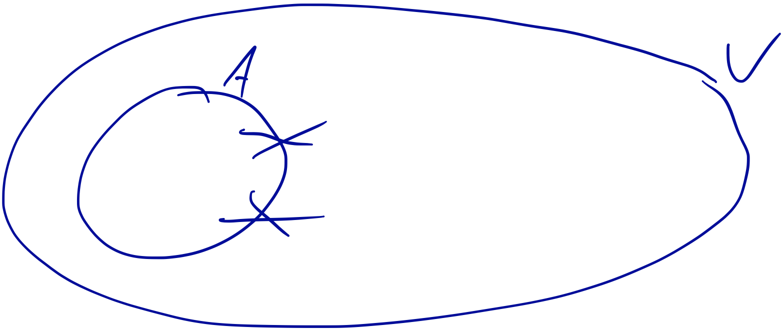


Correctness of the MST Algorithms

Lemma 6.3.

A graph $G = (V, E)$ is connected if and only if there is no set $A \subseteq V$, $\emptyset \neq A \neq V$, with $\delta(A) = \emptyset$.



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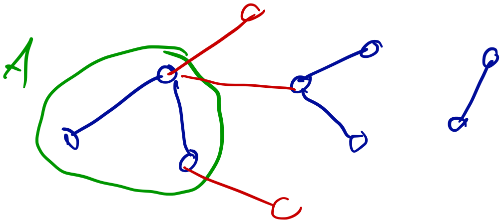

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- Correctness of Prim's Algorithm immediately follows.
- Kruskal: Whenever an edge $e = \{v, w\}$ is added, it is cheapest edge in cut induced by subset of nodes currently reachable from v.

Efficiency of Prim's Algorithm

Prim's Algorithm for MST

1. Set $U := \{r\}$ for some node $r \in V$ and $F := \emptyset$; set $T := (U, F)$.
2. While $U \neq V$, determine a minimum cost edge $e \in \delta(U)$.
3. Set $F := F \cup \{e\}$ and $U := U \cup \{w\}$ with $e = \{v, w\}$, $w \in V \setminus U$.

- Straightforward implementation achieves running time $O(nm)$ where, as usual, $n := |V|$ and $m := |E|$:
  - the while-loop has $n - 1$ iterations;
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- Best known running time is $O(m + n \log n)$ (uses Fibonacci heaps).

Efficiency of Kruskal's Algorithm

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1. Sort the edges in E such that $c(e_1) \leq c(e_2) \leq \cdots \leq c(e_m)$.
2. Set $T := (V, \emptyset)$.
3. For $i := 1$ to m do:
   If adding e_i to T does not create a circuit, then add e_i to T.

Theorem 6.5.

Kruskal's Algorithm can be implemented to run in $O(m \log m)$ time.

Minimum Spanning Trees and Linear Programming

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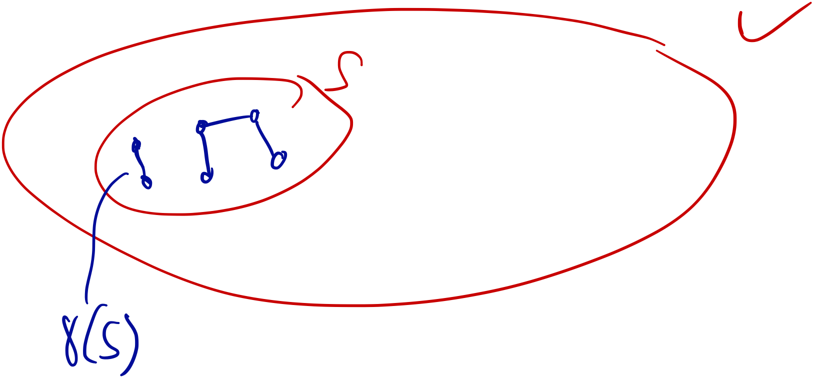
- For $S \subseteq V$ let $\gamma(S) := \{e = \{v, w\} \in E \mid v, w \in S\}$.



Minimum Spanning Trees and Linear Programming

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- For $S \subseteq V$ let $\gamma(S) := \big\{e = \{v, w\} \in E \mid v, w \in S\big\}$.
- For a vector $x \in \mathbb{R}^E$ and a subset $B \subseteq E$ let $x(B) := \sum_{e \in B} x_e$.