Minimum Spanning Trees and Linear Programming

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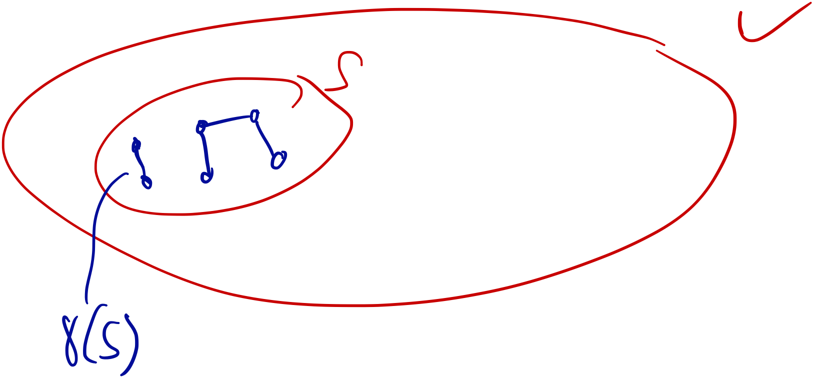
- For $S \subseteq V$ let $\gamma(S) := \{e = \{v, w\} \in E \mid v, w \in S\}$.



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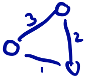
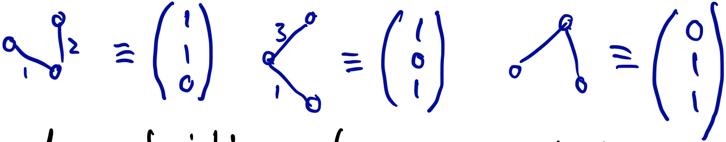
- F does not contain circuit due to (6.1) and $n - 1$ edges due to (6.2).
- Thus, F forms a spanning tree of G.
- Moreover, the edge set of an arbitrary spanning tree of G yields a feasible solution $x \in \{0, 1\}^E$.

Discrete Problems as geometric problems:

Graph G:

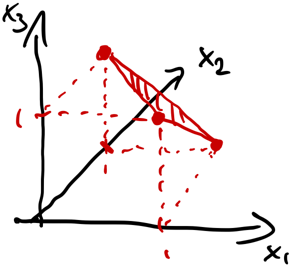


Spanning trees of G as characteristic vectors



Convex hull of characteristic vectors = polytope

Computing a MST = linear optimisation over this polytope



Question: How to obtain a description of this polytope by linear constraints.

We solve this here for the MST problem.

Minimum Spanning Trees and Linear Programming (cont.)

Consider LP relaxation of the integer programming formulation:

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Theorem 6.6.

Let $x^* \in \{0,1\}^E$ be the characteristic vector of an MST. Then x^* is an optimal solution to the LP above.

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Corollary 6.7.

The vertices of the polytope given by the set of feasible LP solutions are exactly the characteristic vectors of spanning trees of G. The polytope is thus the convex hull of the characteristic vectors of all spanning trees.

by definition of vertex

Proof of Thm 6.6:

We compute a MST with Kruskal and show that its characteristic vector x^* is an optimal solution of the LP.

Idea

Also construct a dual solution p with Kruskal and show that the complementary slackness conditions are fulfilled.

Primal (P):

$$\begin{aligned} \min \quad & c^T \cdot x \\ \text{s.t.} \quad & x(\gamma(S)) \leq |S| - 1 \qquad && \forall \emptyset \neq S \subset V \quad \to \text{dual variables } z_S \\ & x(E) = |V| - 1 && \to \text{dual variable } z_V \\ & x_e \geq 0 && \forall e \in E \end{aligned}$$

Dual (D):

$$\begin{aligned} \max \quad & \sum_{\emptyset \neq S \subseteq V} (|S|-1) \cdot z_S \\ \text{s.t.} \quad & \sum_{S : e \in \gamma(S)} z_S \leq c_e && \forall e \\ & z_S \leq 0 && \forall \emptyset \neq S \subsetneq V \\ & z_V \text{ free} \end{aligned}$$

$[p_S := -z_S]$

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