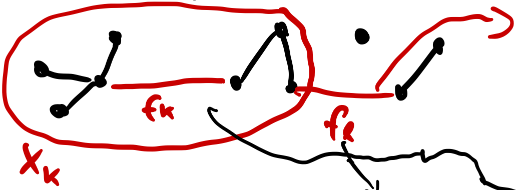
Proof of Thm 6.6 (cont.):
(1) Construct dual solution:

- Kruskal constructs MST T with edge set
  - $E(T) = \{f_1, f_2, \ldots, f_{n-1}\}$, and $c(f_1) \leq c(f_2) \leq \ldots \leq c(f_{n-1})$
- Every edge f_k creates a new connected component $X_k \subseteq V$ by joining two smaller connected components. Note that $X_{n-1} = V$.



f_ℓ: first edge after f_k leaving X_k, i.e. $f_\ell \in \delta(X_k)$

Define:

$$p_{X_k} = c_e(f_\ell) - c_e(f_k) \geq 0$$

$$p_V = -c(f_{n-1})$$

$$p_S = 0 \text{ for all other } S \subseteq V$$

Proof of Thm 6.6 (cont.):

(2) Show that p is feasible for the dual:

- Sign constraints fulfilled by construction.

$$\min \sum_{\emptyset \neq S \subseteq V} (|S|-1)p_S$$

$$\text{s.t.} \quad \sum_{S: e \in \gamma(S)} p_S \geq -c(e) \qquad \forall e \in E$$

$$p_S \geq 0 \qquad \forall \emptyset \neq S \subset V$$

$$p_V \text{ free}$$

$$\sum_{S: e \in \gamma(S)} p_S = \sum_{k: e \in \gamma(X_k)} p_{X_k}$$

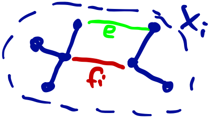
$$= p_{X_i} + p_{X_{\ell_1}} + p_{X_{\ell_2}} + \dots + p_V$$

← sets containing e growing w.r.t. $\subseteq$
let X_i be the smallest such sets.

$$= [c(f_{\ell_1}) - c(f_i)] + [c(f_{\ell_2}) - c(f_{\ell_1})] + \dots + [-c(f_{n-1})]$$

$$= -c(f_i)$$

$$\geq -c(e)$$



either $e = f_i$ (i.e. e will be in MST)
$\Rightarrow c(e) = c(f_i)$
or $e \neq f_i$
$\Rightarrow c(e) \geq c(f_i)$

$\Rightarrow$ dual constraints are fulfilled

$\Rightarrow$ p is dual feasible

Proof of Thm 6.6 (cont.):

(3) Show that x^* and p fulfill complementary slackness conditions:

- $x_e^* > 0 \quad (e \in E(T))$

$$\Rightarrow \sum_{S : e \in \delta(S)} p_S = -c(e)$$

- $p_S \neq 0 \Rightarrow S = X_k$ for some k

$\overset{\text{Def } X_k}{\Rightarrow}$ S = vertex set of tree edges that form a subtree of the final MST

$$\Rightarrow x(\delta(S)) = |S| - 1$$

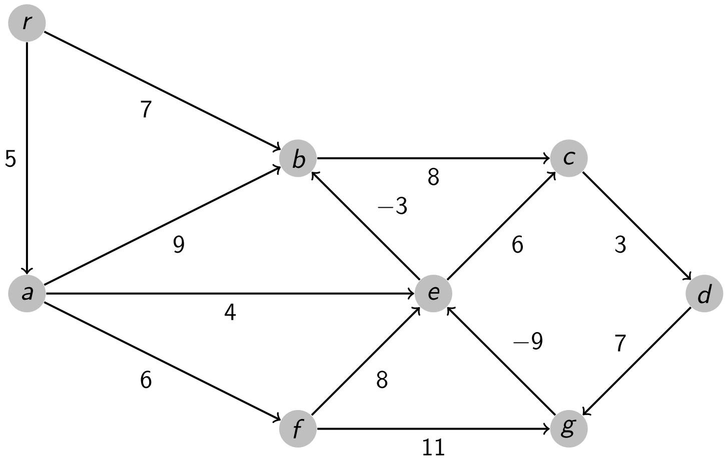
$\Rightarrow$ primal slack $= 0$

$\Rightarrow$ p and x^* are optimal

Shortest Path Problem

Given: digraph $D = (V, A)$, node $r \in V$, arc costs c_a, $a \in A$.

Task: for each $v \in V$, find dipath from r to v of least cost (if one exists)



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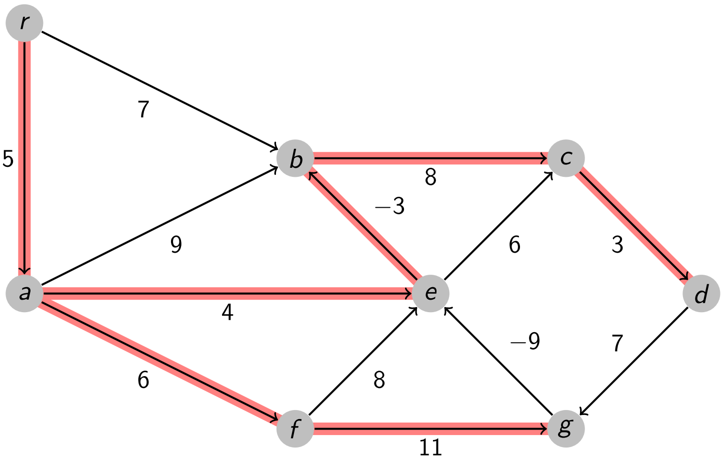

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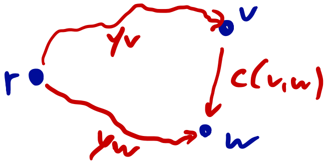
Basic idea behind all algorithms for solving shortest path problem:

If y_v, $v \in V$, is the least cost of a dipath from r to v, then

$$\boxed{y_v + c_{(v,w)} \geq y_w} \quad \text{for all } (v, w) \in A. \tag{6.3}$$



↳ triangular inequality