Shortest Path Problem

Given: digraph $D = (V, A)$, node $r \in V$, arc costs c_a, $a \in A$.

Task: for each $v \in V$, find dipath from r to v of least cost (if one exists)

Remarks:

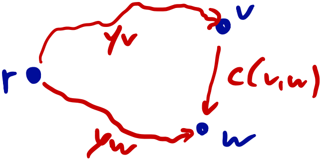
- Existence of r-v-dipath can be checked, e. g., by breadth-first search.
- Ensure existence of r-v-dipaths: add arcs (r, v) of suffic. large cost.

Basic idea behind all algorithms for solving shortest path problem:

If y_v, $v \in V$, is the least cost of a dipath from r to v, then

$$y_v + c_{(v,w)} \geq y_w \quad \text{for all } (v, w) \in A. \tag{6.3}$$

↳ triangular inequality



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Remarks:

- More generally, subpaths of shortest paths are shortest paths!
- If there is a shortest r-v-dipath for all $v \in V$, then there is a shortest path tree, i. e., a directed spanning tree T rooted at r such that the unique r-v-dipath in T is a least-cost r-v-dipath in D.

Feasible Potentials

Definition 6.8.

A vector $y \in \mathbb{R}^V$ is a feasible potential if it satisfies (6.3).

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Corollary 6.10.

If y is a feasible potential with $y_r = 0$ and P an r-v-dipath of cost y_v, then P is a least-cost r-v-dipath. □

Ford's Algorithm

Ford's Algorithm

- i Set $y_r := 0$, $p(r) := r$, $y_v := \infty$, and $p(v) :=$ null, for all $v \in V \setminus \{r\}$.
- ii While there is an arc $a = (v, w) \in A$ with $y_w > y_v + c_{(v,w)}$, set

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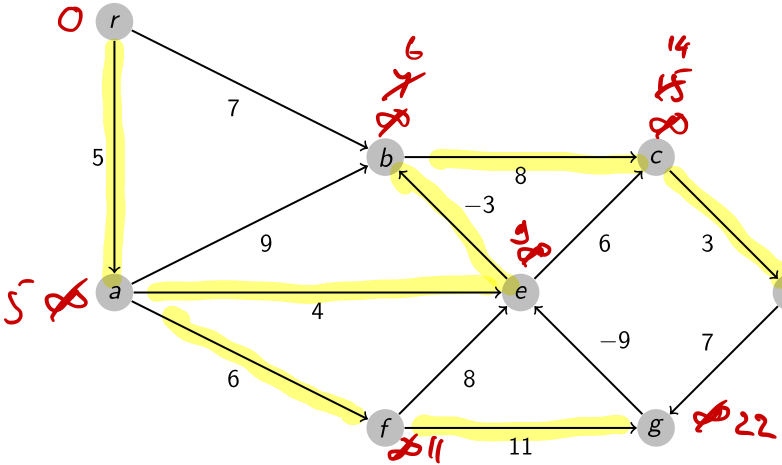

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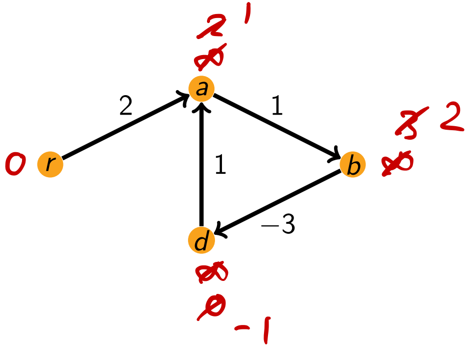

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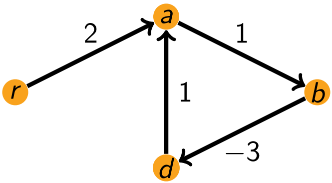

Observation:
The algorithm does not terminate because of the negative-cost dicircuit.

Validity of Ford's Algorithm

Lemma 6.11.

If there is no negative-cost dicircuit, then at any stage of the algorithm:

a. if $y_v \neq \infty$, then y_v is the cost of some simple dipath from r to v;
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Theorem 6.12.

If there is no negative-cost dicircuit, then Ford's Algorithm terminates after a finite number of iterations. At termination, y is a feasible potential with $y_r = 0$ and, for each node $v \in V$, p defines a least-cost r-v-dipath.

Feasible Potentials and Negative-Cost Dicircuits

Theorem 6.13.

A digraph $D = (V, A)$ with arc costs $c \in \mathbb{R}^A$ has a feasible potential if and only if there is no negative-cost dicircuit.

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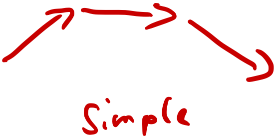

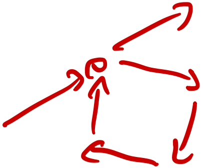

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Lemma 6.14.

If c is integer-valued, $C := 2 \max_{a \in A} |c_a| + 1$, and there is no negative-cost dicircuit, then Ford's Algorithm terminates after at most $C\, n^2$ iterations.

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Proof: Exercise. □

Feasible Potentials and Linear Programming

As a consequence of Ford's Algorithm we get:

Theorem 6.15.

Let $D = (V, A)$ be a digraph, $r, s \in V$, and $c \in \mathbb{R}^A$. If, for every $v \in V$, there exists a least-cost dipath from r to v, then

$$\min\{c(P) \mid P \text{ an } r\text{-}s\text{-dipath}\} = \max\{y_s - y_r \mid y \text{ a feasible potential}\} \ .$$

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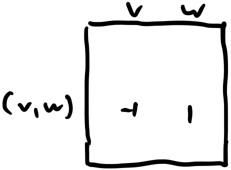
Formulate the right-hand side as a linear program and consider the dual:

$$\begin{aligned} \max \quad & y_s - y_r \\ \text{s.t.} \quad & y_w - y_v \leq c_{(v,w)} \\ & \quad \text{for all } (v, w) \in A \end{aligned}$$



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$$\begin{aligned} \min \quad & c^T \cdot x \\ \text{s.t.} \quad & \sum_{a \in \delta^-(v)} x_a - \sum_{a \in \delta^+(v)} x_a = b_v \quad \forall v \in V \\ & x_a \geq 0 \qquad \text{for all } a \in A \end{aligned}$$

with $b_s = 1$, $b_r = -1$, and $b_v = 0$ for all $v \notin \{r, s\}$.