Feasible Potentials and Linear Programming

As a consequence of Ford's Algorithm we get:

Theorem 6.15.

Let $D = (V, A)$ be a digraph, $r, s \in V$, and $c \in \mathbb{R}^A$. If, for every $v \in V$, there exists a least-cost dipath from r to v, then

$$\min\{c(P) \mid P \text{ an } r\text{-}s\text{-dipath}\} = \max\{y_s - y_r \mid y \text{ a feasible potential}\} \ .$$

Formulate the right-hand side as a linear program and consider the dual:

$$\begin{aligned} \max \quad & y_s - y_r \\ \text{s.t.} \quad & y_w - y_v \leq c_{(v,w)} \\ & \text{for all } (v, w) \in A \end{aligned}$$

$$\begin{aligned} \min \quad & c^T \cdot x \\ \text{s.t.} \quad & \sum_{a \in \delta^-(v)} x_a - \sum_{a \in \delta^+(v)} x_a = b_v \quad \forall v \in V \\ & x_a \geq 0 \qquad \text{for all } a \in A \end{aligned}$$

with $b_s = 1$, $b_r = -1$, and $b_v = 0$ for all $v \notin \{r, s\}$.

Notice: The dual is the LP relaxation of an ILP formulation of the shortest r-s-dipath problem ($x_a \hat{=}$ number of times a shortest r-s-dipath uses arc a).

Bases of Shortest Path LP

Consider again the dual LP:

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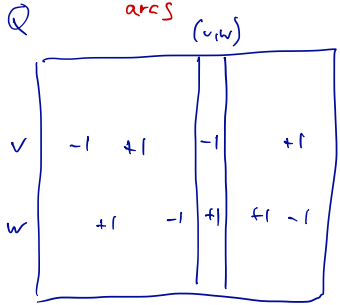
The underlying matrix Q is the incidence matrix of D.



called the arc-node incident matrix

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Lemma 6.16.

Let $D = (V, A)$ be a connected digraph and Q its incidence matrix. A subset of columns of Q indexed by a subset of arcs $F \subseteq A$ forms a basis of the linear subspace of $\mathbb{R}^n$ spanned by the columns of Q if and only if F is the arc-set of a spanning tree of D.

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Proof: Exercise. □

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Ford's Algorithm

i Set $y_r := 0$, $p(r) := r$, $y_v := \infty$, and $p(v) := \text{null}$, for all $v \in V \setminus \{r\}$.

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Goal: Find short sequence $\mathcal{S}$ such that, for all $v \in V$, a least-cost r-v-dipath is embedded in $\mathcal{S}$.

Ford-Bellman Algorithm

Basic idea:

- Every simple dipath is embedded in $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_{n-1}$ where, for all i, $\mathcal{S}_i$ is an ordering of A.

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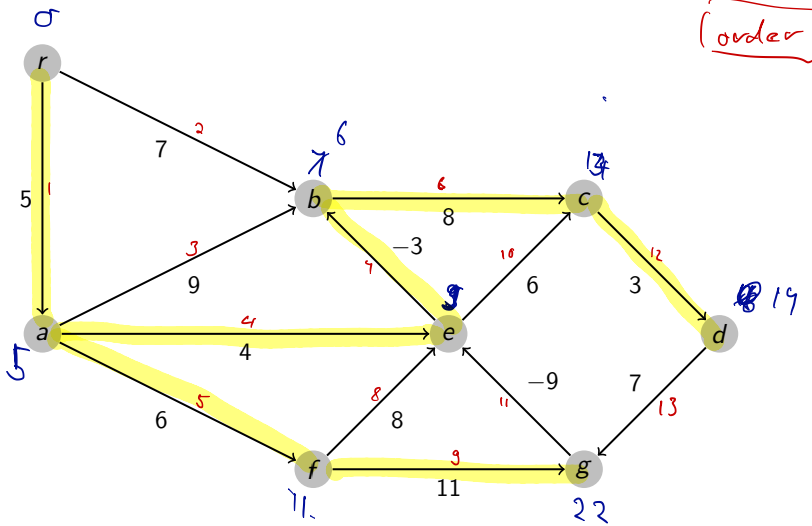
Theorem 6.18.

The algorithm runs in $O(nm)$ time. If, at termination, y is a feasible potential, then p yields a least-cost r-v-dipath for each $v \in V$. Otherwise, the given digraph contains a negative-cost dicircuit. □

Ford-Bellman Example



Acyclic Digraphs and Topological Orderings

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a. An ordering $v_1, v_2, \ldots, v_n$ of V so that $i < j$ for each $(v_i, v_j) \in A$ is called a topological ordering.

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Theorem 6.20.

The shortest path problem on acyclic digraphs can be solved in time $O(m)$.

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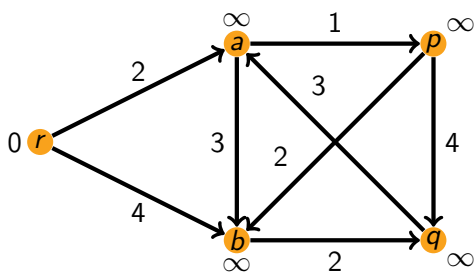

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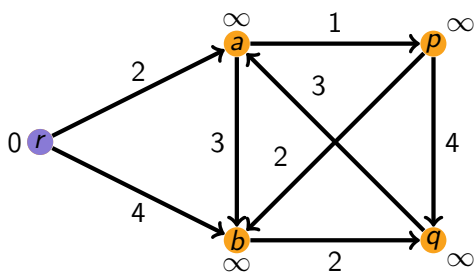

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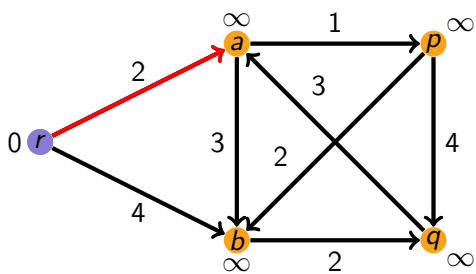

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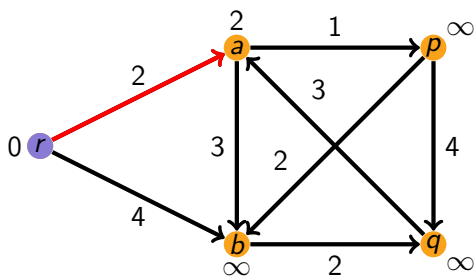

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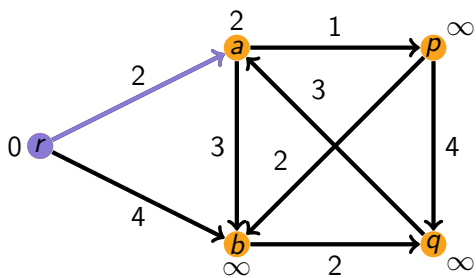

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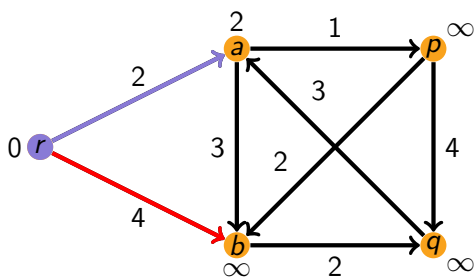

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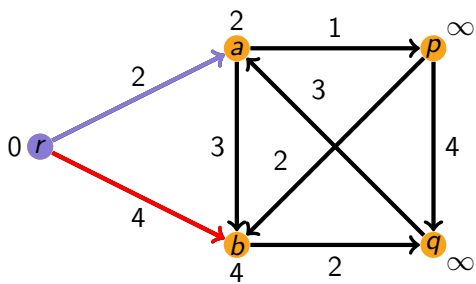

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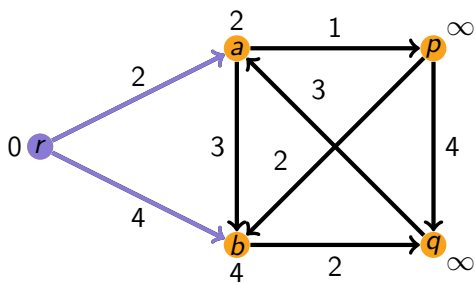

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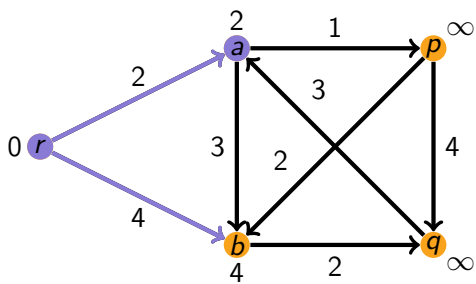

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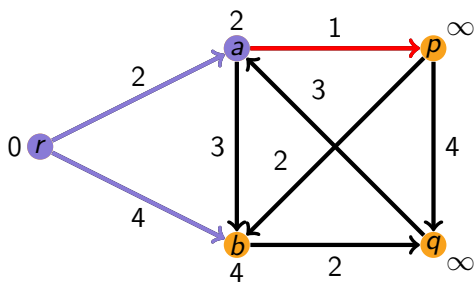

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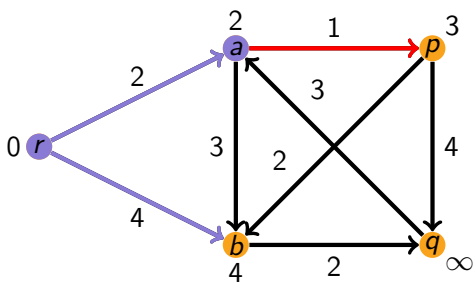

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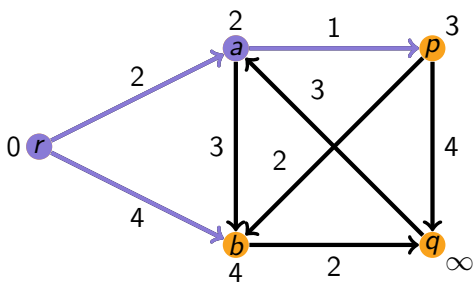

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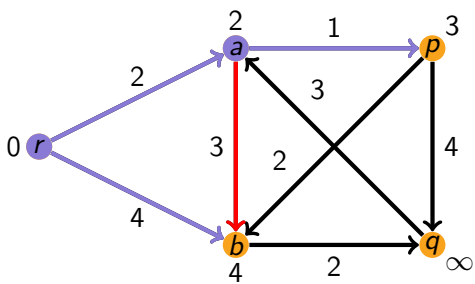

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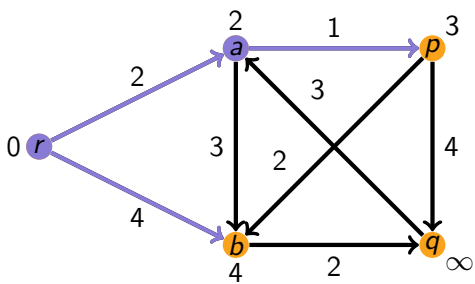

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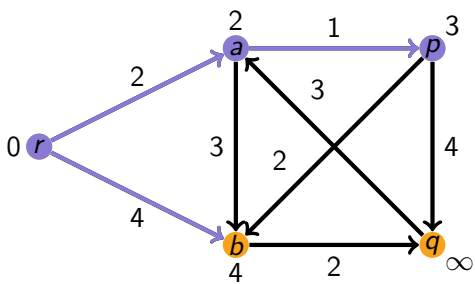

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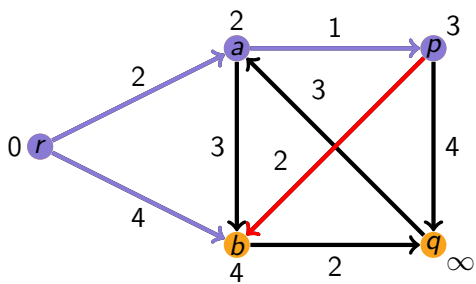

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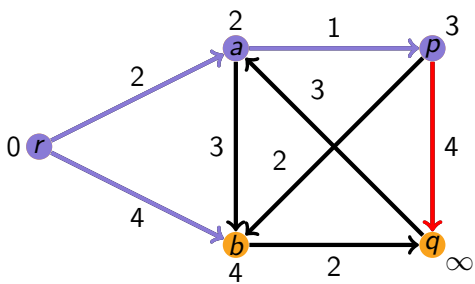

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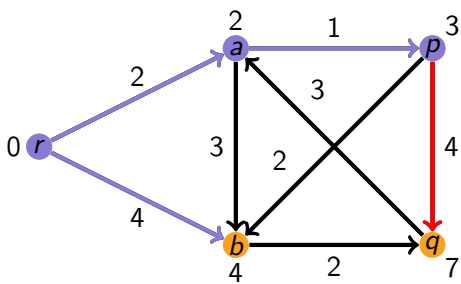

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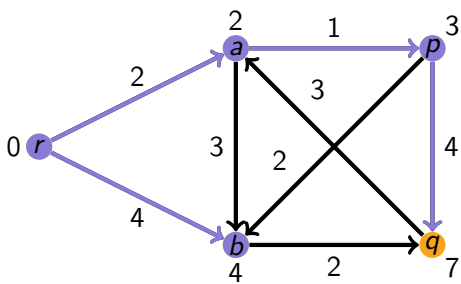

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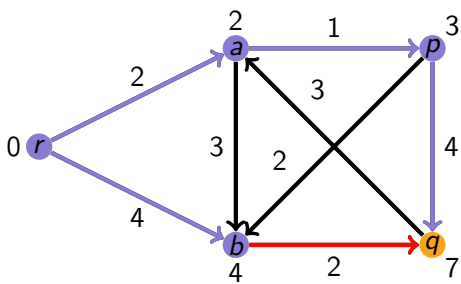

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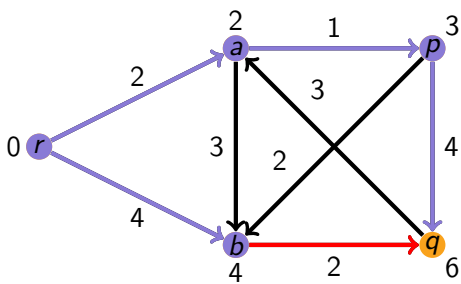

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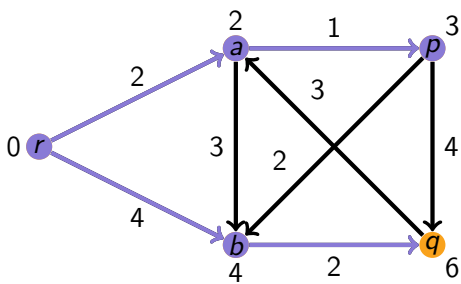

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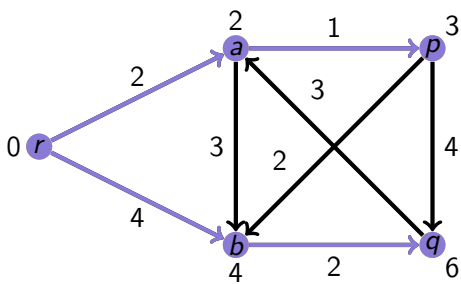

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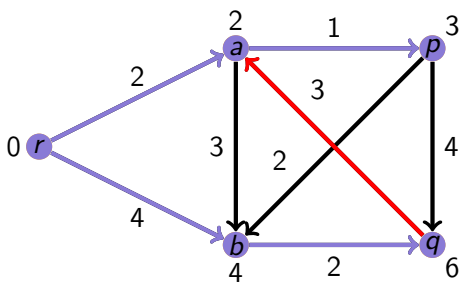

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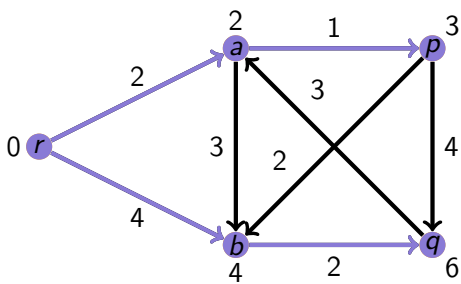

Correctness of Dijkstra's Algorithm

Lemma 6.21.

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If u is deleted from S before v, then $y'_u \leq y'_v$.

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Remark: The for-loop in Dijkstra's Algorithm (step iv) can be modified such that only arcs (v, w) with $w \in S$ are considered.