# Chapter 7: Maximum Flow Problems

(cp. Cook, Cunningham, Pulleyblank & Schrijver, Chapter 3)

# Maximum $s$-$t$-Flow Problem

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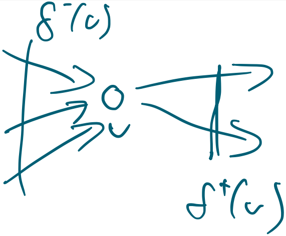

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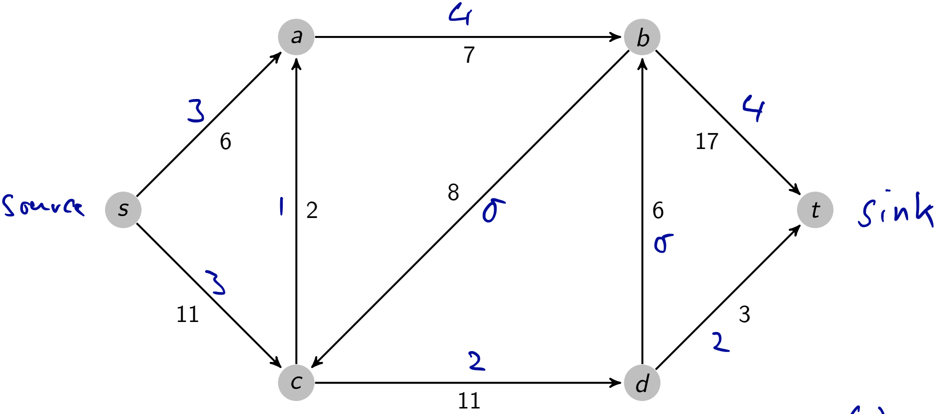
The maximum $s$-$t$-flow problem asks for a feasible $s$-$t$-flow in $D$ of maximum value.

# Example




$ex_x(s) = -6$

$ex_x(t) = 6$

# $s$-$t$-Flows and $s$-$t$-Cuts

For a subset of nodes $U \subseteq V$, the excess of $U$ is defined as

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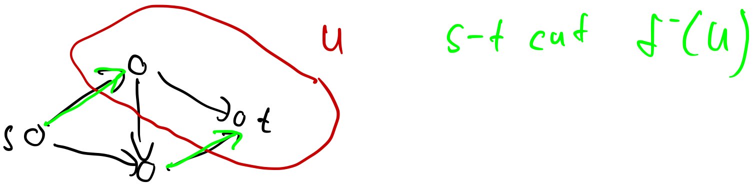
For $U \subseteq V \setminus \{s\}$ with $t \in U$, the subset of arcs $\delta^-(U)$ is called an $s$-$t$-cut.



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**Lemma 7.3.**

Let $U \subseteq V \setminus \{s\}$ with $t \in U$. The value of a feasible $s$-$t$-flow $x$ is at most the capacity $u(\delta^-(U))$ of the $s$-$t$-cut $\delta^-(U)$. Equality holds if and only if $x_a = u_a$ for each $a \in \delta^-(U)$ and $x_a = 0$ for each $a \in \delta^+(U)$.

Lemma 7.3:

$$\text{flow value} = ex_x(U)$$

$$= x(\delta^-(U)) - \underbrace{x(\delta^+(U))}_{\geq 0}$$

$$\leq x(\delta^-(U))$$

$$\leq u(\delta^-(U))$$

# Residual Graph and Residual Arcs

For $a = (v, w) \in A$, let $a^{-1} := (w, v)$ be the corresponding backward arc and $A^{-1} := \{a^{-1} \mid a \in A\}$.

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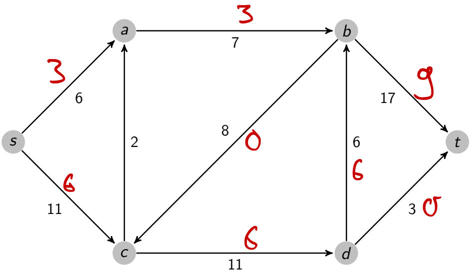
- For a feasible flow $x$, the set of residual arcs is given by

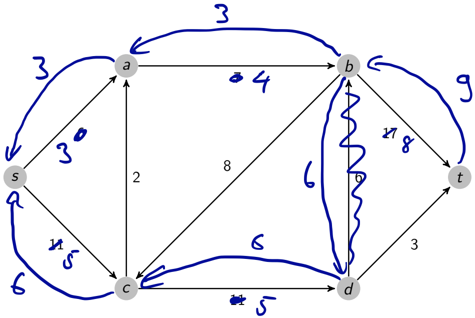
$$A_x := \{a \in A \mid x_a < u_a\} \cup \{a^{-1} \in A^{-1} \mid x_a > 0\} \ .$$

- For $a \in A$, define the residual capacity $u_x(a)$ as

$$u_x(a) := u(a) - x(a) \quad \text{if } a \in A_x, \quad \text{and} \quad u_x(a^{-1}) := x(a) \quad \text{if } a^{-1} \in A_x.$$

- The digraph $D_x := (V, A_x)$ is called the residual graph of $x$.



# $x$-augmenting paths

Observation:

- If $x$ is a feasible flow in $(D, u)$ and $y$ a feasible flow in $(D_x, u_x)$, then

$$z(a) := x(a) + y(a) - y(a^{-1}) \quad \text{for } a \in A$$

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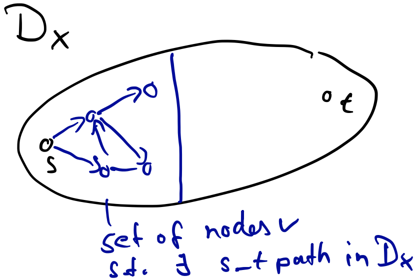
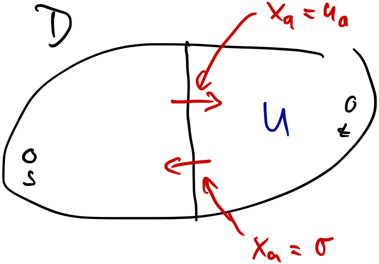
**Lemma 7.4.**
If $x$ is a feasible $s$-$t$-flow such that $D_x$ does not contain an $s$-$t$-dipath, then $x$ is a maximum $s$-$t$-flow.



# Max-Flow Min-Cut Theorem and Ford-Fulkerson Algorithm

**Theorem 7.5 (Max-Flow Min-Cut Theorem).**

The maximum $s$-$t$-flow value equals the minimum capacity of an $s$-$t$-cut.

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**Corollary.**

A feasible $s$-$t$-flow $x$ is maximum if and only if $D_x$ does not contain an $s$-$t$-dipath. □

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The maximum $s$-$t$-flow value equals the minimum capacity of an $s$-$t$-cut.

## Corollary.

A feasible $s$-$t$-flow $x$ is maximum if and only if $D_x$ does not contain an $s$-$t$-dipath. □

## Ford-Fulkerson Algorithm

i set $x := 0$;

ii while there is an $s$-$t$-dipath $P$ in $D_x$

iii set $x := x + \delta \cdot \chi^P$ with $\delta := \min\{u_x(a) \mid a \in P\}$;

Here, $\chi^P : A \to \{0, 1, -1\}$ is the characteristic vector of dipath $P$ defined by

$$\chi^P(a) = \begin{cases} 1 & \text{if } a \in P, \\ -1 & \text{if } a^{-1} \in P, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for all } a \in A.$$

# Ford-Fulkerson Example