Termination of the Ford-Fulkerson Algorithm

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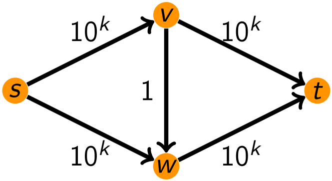
When an arbitrary x-augmenting path is chosen in every iteration, the Ford-Fulkerson Algorithm can behave badly:



Running Time of the Ford-Fulkerson Algorithm

Theorem 7.7.

If all capacities are integral and the maximum flow value is $K < \infty$, then the Ford-Fulkerson Algorithm terminates after at most K iterations. Its running time is $O(m \cdot K)$ in this case.

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A variant of the Ford-Fulkerson Algo. is the Edmonds-Karp Algorithm:

- In each iteration, choose shortest s-t-dipath in D_x (edge lengths=1)

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Theorem 7.8.

The Edmonds-Karp Algorithm terminates after at most $n \cdot m$ iterations; its running time is $O(n \cdot m^2)$.

Remark: The Edmonds-Karp Algorithm can be implemented with running time $O(n^2 \cdot m)$.

Arc-Based LP Formulation

Straightforward LP formulation of the maximum s-t-flow problem:

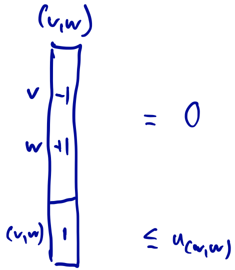
$$\max \sum_{a \in \delta^+(s)} x_a - \sum_{a \in \delta^-(s)} x_a \qquad \text{net flow out of } s$$

$$\text{s.t.} \quad \sum_{a \in \delta^-(v)} x_a - \sum_{a \in \delta^+(v)} x_a = 0 \qquad \text{for all } v \in V \setminus \{s,t\} \quad \text{(flow conservation)} \quad y_v$$

$$x_a \le u(a) \qquad \text{for all } a \in A \quad \text{(capacity constraints)} \quad z_a$$

$$x_a \ge 0 \qquad \text{for all } a \in A$$

Column of primal



Define $y_s = 1$, $y_t = 0$

$$\min \sum_{a \in A} u(a) \cdot z_a$$

$$\begin{aligned}
\text{s.t.} \quad -y_v + y_w + z_{vw} &\ge 0 && \forall (v,w) \in A : v,w \in V \setminus \{s,t\} \\
y_w + z_{sw} &\ge 1 && \forall (s,w) \in A \\
-y_v + z_{vs} &\ge -1 && \forall (v,s) \in A \\
-y_v + z_{vt} &\ge 0 && \forall (v,t) \in A \\
y_w + z_{tw} &\ge 0 && \forall (t,w) \in A \\
z_a &\ge 0 && \forall a \in A
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& x_a \le u(a) && \text{for all } a \in A \\
& x_a \ge 0 && \text{for all } a \in A
\end{aligned}$$

Dual LP:

$$\begin{aligned}
\min \quad & \sum_{a \in A} u(a) \cdot z_a \\
\text{s.t.} \quad & y_w - y_v + z_{(v,w)} \ge 0 && \text{for all } (v, w) \in A \\
& y_s = 1, \quad y_t = 0 \\
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Dual Solutions and s-t-Cuts

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Observation: An s-t-cut $\delta^+(U)$ (with $U \subseteq V \setminus \{t\}$, $s \in U$) yields feasible dual solution (y, z) of value $u(\delta^+(U))$:

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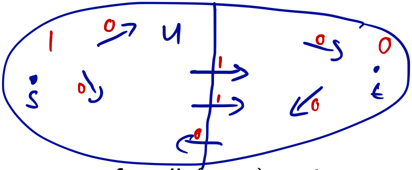
Dual Solutions and s-t-Cuts



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- let z be the characteristic vector $\chi^{\delta^+(U)}$ of $\delta^+(U)$ (i. e., $z_a = 1$ for $a \in \delta^+(U)$, $z_a = 0$ for $a \in A \setminus \delta^+(U)$)

Dual Solutions and s-t-Cuts

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Theorem 7.9.

There exists an s-t-cut $\delta^+(U)$ (with $U \subseteq V \setminus \{t\}$, $s \in U$) such that the corresponding dual solution (y, z) is an optimal dual solution.

Proof Thm 4.9:

<table>
<tr><td>opt value of dual</td><td></td><td>min cap of a cut</td></tr>
<tr><td>|| strong duality</td><td></td><td>|| max-flow min-cut theorem</td></tr>
<tr><td>opt value of primal</td><td>=</td><td>max flow value</td></tr>
</table>