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UNIVERSITY OF LIVERPOOL

First Semester Examinations 2014/15

OPTIMISATION

TIME ALLOWED : Two and a Half Hours

INSTRUCTIONS TO CANDIDATES

Answer **ALL** questions.

THIS PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

1. **LP-Basics / Geometry**

(a) Formally define the terms *polyhedron* and *halfspace*. How do those geometric objects relate to each other? **[6 marks]**

(b) Among the vectors $(1,0,1)$, $(-1,1,2)$, $(-1,-2,5)$, $(0,3,1)$ find a maximum size subset of linearly independent vectors and provide an argument that they indeed are linearly independent. Is this subset unique ? **[6 marks]**

(c) Consider the following LP

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & A \cdot x = b \\ & x \geq 0, \end{aligned}$$

where

$$A = \begin{pmatrix} 2 & 0 & 0 & 3 & 1 & 0 \\ -2 & 2 & 3 & -2 & 2 & 1 \\ 1 & -1 & -1 & 1 & 0 & 1 \\ -2 & 2 & 2 & -1 & 1 & 0 \end{pmatrix},$$

$b = (2,3,1,-1)^T$, and $c^T = (0,4,2,1,1,0)$.
Let M be the basis consisting of the first 4 columns of A. Compute M^{-1}. **[7 marks]**

(d) Now let B consists of the last 4 columns of A. You are given

$$B^{-1} = \begin{pmatrix} -1/2 & -2 & 2 & 9/2 \\ 0 & -1 & 1 & 3 \\ 1 & 3 & -3 & -6 \\ -1/2 & -1 & 2 & 5/2 \end{pmatrix}.$$

Compute the corresponding basic primal and dual solutions. Is the primal basic solution feasible? Why or why not? **[6 marks]**

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2. **Simplex**

Consider the following linear program:

$$\begin{array}{rrcrcrcl}
\min & -3x_1 & - & 5x_2 & - & 2x_3 & & \\
s.t. & x_1 & + & x_2 & + & 2x_3 & \leq & 2 \\
 & 2x_1 & + & 3x_2 & + & x_3 & \leq & 5 \\
 & 2x_1 & + & 4x_2 & + & 2x_3 & \leq & 8 \\
 & & & x_1, & & x_2, \quad x_3 & \geq & 0
\end{array}$$

(a) Convert the problem into standard form and construct a basic feasible solution at which $(x_1, x_2, x_3) = (0, 0, 0)$. **[3 marks]**

(b) Carry out the full tableau implementation of the simplex method, starting at the basic feasible solution of part (a). Use Bland's rule to determine the pivot element. In every step mark the pivot element and provide the current values of the objective function and the variables. **[15 marks]**

(c) We want to solve the linear program

$$\begin{array}{rrcrcrl}
\min & 4x_1 & + & 4x_2 & + & x_3 & \\
s.t. & x_1 & + & x_2 & + & x_3 & = 2 \\
 & 2x_1 & + & x_2 & & & = 3 \\
 & 3x_1 & + & 2x_2 & + & x_3 & = 5 \\
 & & & x_1, & & x_2, \quad x_3 & \geq 0
\end{array}$$

using the Big-M method. Set up the initial simplex tableau and do the required operations that compute the reduced costs. In the resulting tableau, mark **all** candidates for the first pivot element. **[7 marks]**

Remark: You **don't** have to perform pivoting steps here. Computing the reduced costs is sufficient.

3. **Duality / Complementary Slackness**

(a) Consider the following pair of primal and dual LPs:

$$\begin{array}{ll} \min & c^T \cdot x \\ \text{s.t.} & A \cdot x \geq b \end{array} \qquad\qquad \begin{array}{ll} \max & p^T \cdot b \\ \text{s.t.} & p^T \cdot A = c^T \\ & p > 0 \end{array}$$

Formulate the *Complementary Slackness Theorem*. Which part of the theorem is trivially fulfilled for the above primal/dual pair? **[5 marks]**

(b) Construct the dual of the following LP **[4 marks]**

$$\begin{array}{lrcrcrcrcl} \min & 4x_1 & - & 3x_2 & & & & & & \\ s.t. & 2x_1 & + & x_2 & - & x_3 & + & x_4 & = & 2 \\ & -4x_1 & + & 3x_2 & - & x_3 & & & \geq & 5 \\ & 2x_1 & - & 6x_2 & + & x_3 & + & 3x_4 & \leq & 8 \\ & & & & & & & x_1 & \leq & 0 \\ & & & & & & x_2, & x_3 & \geq & 0 \end{array}$$

(c) Consider the primal problem

$$\begin{array}{ll} \min & c^T \cdot x \\ \text{s.t.} & A \cdot x \geq b \\ & x \geq 0 \end{array}$$

Form the dual problem and convert it into an equivalent minimisation problem. Derive a set of conditions on the matrix A and the vectors b, c, under which the dual is identical to the primal. Construct an example in which these conditions are satisfied. **[8 marks]**

(d) Consider the following linear program:

$$\begin{array}{lrcrcrcrcr} \min & x_1 & + & 5x_2 & + & 2x_3 & + & 2x_4 & & \\ s.t. & x_1 & & & & & + & x_4 & = & 1 \\ & x_1 & & & + & x_3 & - & 5x_4 & \leq & 6 \\ & 2x_1 & + & x_2 & + & x_3 & + & x_4 & = & 10 \\ & 2x_1 & + & x_2 & & & + & 5x_4 & = & 6 \\ & & & x_2 & + & x_3 & + & x_4 & \geq & 5 \\ & & & & x_1, & x_2, & x_3, & x_4 & \geq & 0 \end{array}$$

Verify that $x^* = (0, 1, 8, 1)^T$ is optimal, using complementary slackness. **[8 marks]**

4. **Trees, Paths, Flows, and Matchings**

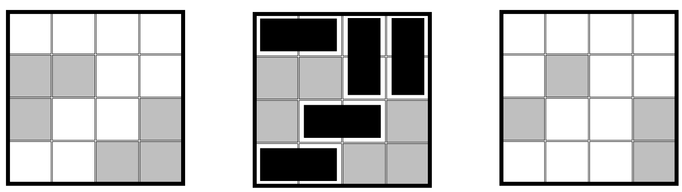
(a) We discussed Prim's and Kruskal's algorithms for computing spanning trees in an undirected graph $G = (V, E)$. Briefly describe both of them. **[8 marks]**

(b) Give an LP formulation for the Maximum s-t-Flow Problem. **[5 marks]**

(c) An $n \times n$ chessboard B is an $n \times n$ array with a set of squares $S = [n] \times [n]$. A pruned $n \times n$ chessboard is a chessboard with a set $X \subset S$ of squares excluded. Two squares are said to be neighbours if they have a common border, e.g. $u = (1, 1)$ has as neighbours the squares $(1, 2)$ and $(2, 1)$, but not the square $(2, 2)$. The neighbours of a square u can be assumed to be given in a set $N(u)$.

The covering problem for pruned chessboards is the problem to cover the unpruned squares $U = S \setminus X$ of the board with 2×1 dominoes (tiles) such that every unpruned square is covered by exactly one tile and no pruned square is covered.

An example is given in the pictures below (Pruned squares are marked grey, dominoes are marked black. Note that dominoes can be used horizontally as well as vertically).



A pruned chessboard | A cover | No cover possible

Formally, the cover problem for pruned chessboards is as follows:

Given: A pruned chessboard with a set $U \subset [n] \times [n]$ of unpruned squares.
Task: A cover of the unpruned squares U with 2×1 tiles, if such a cover exists.

(i) Model the covering problem for pruned chessboards as a problem to compute a maximum matching in a (bipartite) graph (define a graph $G = (V, E)$). **[4 marks]**

(ii) Formulate the covering problem for pruned chessboards as a linear program. Explicitly state the condition for an instance (n, U) to admit a cover (depending on the solution of your linear program). **[8 marks]**