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UNIVERSITY OF LIVERPOOL

First Semester Examinations 2015/16

OPTIMISATION

TIME ALLOWED : Two and a Half Hours

INSTRUCTIONS TO CANDIDATES

Answer **ALL** questions.

THIS PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

1. **LP-Basics / Geometry**

(a) Formally define the terms *polyhedron* and *halfspace*. How do those geometric objects relate to each other? **[6 marks]**

(b) Among the vectors $(1, 0, -1)$, $(3, -1, 2)$, $(2, -1, 3)$, $(0, 1, 3)$ find a maximum size subset of linearly independent vectors and provide an argument that they indeed are linearly independent. Is this subset unique ? **[6 marks]**

(c) Consider the following LP

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & A \cdot x = b \\ & x \geq 0, \end{aligned}$$

where

$$A = \begin{pmatrix} 2 & 0 & 0 & 2 & 1 & 1 \\ -1 & 2 & 1 & -2 & 2 & 3 \\ -1 & 2 & 2 & 0 & 1 & -2 \\ 1 & -2 & -1 & 1 & 0 & 1 \end{pmatrix},$$

$b = (3, -1, 2, 0)^T$, and $c^T = (0, 4, 10, 5, 5, 0)$.
Let M be the basis consisting of the first 4 columns of A. Compute M^{-1}. Show all work. **[7 marks]**

(d) Now let B consists of the last 4 columns of A. You are given

$$B^{-1} = \begin{pmatrix} 9/5 & -1/5 & -7/5 & -4 \\ 4/5 & -1/5 & -2/5 & -1 \\ -8/5 & 2/5 & 9/5 & 4 \\ 1 & 0 & -1 & -2 \end{pmatrix}.$$

Compute the corresponding basic primal and dual solutions. Is the primal basic solution feasible? Why or why not? **[6 marks]**

2. **Simplex**

Consider the following linear program:

$$\begin{array}{rrcrcrcl} \min & -3x_1 & - & 2x_2 & - & x_3 & & \\ s.t. & x_1 & + & 2x_2 & + & 3x_3 & \leq & 4 \\ & 2x_1 & + & 2x_2 & + & x_3 & \leq & 5 \\ & 2x_1 & + & x_2 & & & \leq & 4 \\ & & & x_1, & & x_2, \quad x_3 & \geq & 0 \end{array}$$

(a) Convert the problem into standard form and construct a basic feasible solution at which $(x_1, x_2, x_3) = (0, 0, 0)$. **[3 marks]**

(b) Carry out the full tableau implementation of the simplex method, starting at the basic feasible solution of part (a). Use Bland's rule to determine the pivot element. In every step mark the pivot element and provide the current values of the objective function and the variables. **[15 marks]**

(c) We want to solve the linear program

$$\begin{array}{rrcrcrl} \min & 3x_1 & + & 3x_2 & + & x_3 & \\ s.t. & 3x_1 & + & 2x_2 & + & x_3 & = 4 \\ & x_1 & + & x_2 & + & x_3 & = 2 \\ & 3x_1 & - & x_2 & & & = 3 \\ & & & x_1, & & x_2, \quad x_3 & \geq 0 \end{array}$$

using the Big-M method. Set up the initial simplex tableau and do the required operations that compute the reduced costs. In the resulting tableau, mark **all** candidates for the first pivot element. **[7 marks]**

Remark: You **don't** have to perform pivoting steps here. Computing the reduced costs is sufficient.

3. **Duality / Complementary Slackness**

(a) Consider the following pair of primal and dual linear programs:

$$\begin{aligned} \min\ & c^T \cdot x \\ \text{s.t.}\ & A \cdot x \geq b \end{aligned} \qquad\qquad \begin{aligned} \max\ & p^T \cdot b \\ \text{s.t.}\ & p^T \cdot A = c^T \\ & p \geq 0 \end{aligned}$$

i. Formally state the *Weak Duality Theorem* using the notation of the above primal-dual pair. **[3 marks]**

ii. Suppose you are given dual feasible solution p. Explain in your own words, what information p provides for the primal problem? **[3 marks]**

(b) Construct the dual of the following linear program: **[5 marks]**

$$\begin{array}{rrcrcrcrcl} \min & 4x_1 & - & 2x_2 & + & x_2 & & & & \\ s.t. & 3x_1 & + & x_2 & - & 2x_3 & + & 2x_4 & = & 2 \\ & -2x_1 & + & 2x_2 & - & x_3 & & & \geq & 5 \\ & 5x_1 & - & 7x_2 & + & x_3 & + & 3x_4 & \leq & 8 \\ & & & & & & x_1, & x_3 & \leq & 0 \\ & & & & & & & x_2 & \geq & 0 \end{array}$$

(c) Consider the following linear program:

$$\begin{array}{rrcrcrcrcr} \min & x_1 & + & 5x_2 & + & 2x_3 & + & 2x_4 & & \\ s.t. & x_1 & & & & & + & x_4 & = & 1 \\ & x_1 & & & + & x_3 & - & 5x_4 & \leq & 6 \\ & 2x_1 & + & x_2 & + & x_3 & + & x_4 & = & 10 \\ & 2x_1 & + & x_2 & & & + & 5x_4 & = & 6 \\ & & & x_2 & + & x_3 & + & x_4 & \geq & 5 \\ & & & x_1, & & x_2, & & x_3, \ x_4 & \geq & 0 \end{array}$$

Verify that $x^* = (0, 1, 8, 1)^T$ is optimal, using complementary slackness. **[8 marks]**

(d) Consider the primal problem

$$\begin{aligned} \min\ & c^T \cdot x \\ \text{s.t.}\ & A \cdot x \geq b \\ & x \geq 0 \end{aligned}$$

Form the dual problem and convert it into an equivalent minimisation problem. Derive a set of conditions on the matrix A and the vectors b, c, under which the dual is identical to the primal. Construct an example in which these conditions are satisfied. **[6 marks]**

4. **Trees, Paths, Flows, and Matchings**

(a) In the lectures, we defined the *shortest path problem* as follows:
Given: digraph $D = (V, A)$, node $r \in V$, arc costs c_a, $a \in A$.
Task: for each $v \in V$, find dipath from r to v of least cost (if one exists)

i. Briefly describe Ford's algorithm. **[5 marks]**

ii. Provide a (minimum) requirement on the digraph $D = (V, A)$, which guarantees that Ford's algorithm terminates. **[3 marks]**

iii. Now consider that we are only interested in the shortest directed path from node r to node s in D. Give an ILP (integer linear program) formulation for the *shortest r-s-dipath problem*. **[5 marks]**

(b) We are given an undirected graph $G = (V, E)$ and we need to test or monitor the edges of the graph. We want to test each edge $\{u, v\} \in E$ exactly $\alpha_{u,v} \geq 0$ times. Each edge $\{u, v\} \in E$ can only be tested by nodes u and v. Every node $u \in V$ can test at most β_u incident edges on each day. All input numbers (α's and β's) are integers and the number of tests that a node performs on each incident edge in any day is also required to be an integer.

The problem is to find a schedule that completes the testing of all edges in the fewest number of days. Show how to model the problem as an ILP (integer linear program). Explain your construction. **[12 marks]**