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UNIVERSITY OF LIVERPOOL

First Semester Examinations 2016/17

OPTIMISATION

TIME ALLOWED : Two and a Half Hours

INSTRUCTIONS TO CANDIDATES

Answer **ALL** questions.

THIS PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

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1. **LP-Basics / Geometry**

   (a) Let $P \subseteq \mathbb{R}^n$ be a polyhedron. Formally define the terms *extreme point* of P and *vertex* of P. How do those geometric objects relate to each other?

   [6 marks]

   (b) Among the vectors $(-1, 0, 1)$, $(2, -1, 3)$, $(3, -1, 2)$, $(3, 1, 0)$ find a maximum size subset of linearly independent vectors and provide an argument that they indeed are linearly independent. Is this subset unique ?

   [6 marks]

   (c) Consider the following linear program

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & A \cdot x = b \\ & x \geq 0, \end{aligned}$$

   where

$$A = \begin{pmatrix} 1 & -1 & -1 & 1 & -1 & -2 \\ 0 & 1 & -2 & -1 & 2 & 2 \\ -2 & 2 & 4 & -1 & 1 & 2 \\ 2 & 1 & 1 & 2 & 0 & 0 \end{pmatrix},$$

   $b = (0, 1, -3, 2)^T$, and $c^T = (1, 0, 12, 6, 2, 0)$.
   Let M be the basis consisting of the first 4 columns of A. Compute M^{-1}. Show all work. **[7 marks]**

   (d) Now let B consist of the last 4 columns of A. You are given

$$B^{-1} = \begin{pmatrix} 1/3 & 0 & 1/3 & 0 \\ -1/6 & 0 & -1/6 & 1/2 \\ 2 & 1 & 1 & 0 \\ -7/4 & -1/2 & -3/4 & 1/4 \end{pmatrix}.$$

   Compute the corresponding basic primal and dual solutions. Is the primal basic solution feasible? Why or why not?

   [6 marks]

2. **Simplex**

Consider the following linear program:

$$\begin{array}{rrcrcrcr} \min & -4x_1 & - & 2x_2 & - & 3x_3 & & \\ s.t. & 2x_1 & + & x_2 & + & x_3 & \leq & 8 \\ & 4x_1 & & & + & 2x_3 & \leq & 10 \\ & 4x_1 & + & x_2 & + & 2x_3 & \leq & 14 \\ & & & x_1, & x_2, & x_3 & \geq & 0 \end{array}$$

(a) Convert the problem into standard form and construct a basic feasible solution at which $(x_1, x_2, x_3) = (0, 0, 0)$. **[3 marks]**

(b) Carry out the full tableau implementation of the simplex method, starting at the basic feasible solution of part (a). Use Bland's rule to determine the pivot element. In every step mark the pivot element and provide the current values of the objective function and the variables. **[15 marks]**

(c) We want to solve the linear program

$$\begin{array}{rrcrcrl} \min & x_1 & + & 5x_2 & - & 4x_3 & \\ s.t. & 4x_1 & + & x_2 & - & x_3 & = 4 \\ & 3x_1 & + & 2x_2 & - & x_3 & = 3 \\ & 5x_1 & - & x_2 & + & 2x_3 & = 6 \\ & & & x_1, & x_2, & x_3 & \geq 0 \end{array}$$

using the Big-M method. Set up the initial simplex tableau and do the required operations that compute the reduced costs. In the resulting tableau, mark **all** candidates for the first pivot element. **[7 marks]**

Remark: You **don't** have to perform pivoting steps here. Computing the reduced costs is sufficient.

3. **Duality / Complementary Slackness**

(a) Consider the following pair of primal and dual linear programs:

$$\begin{array}{ll} \min & c^T \cdot x \\ \text{s.t.} & A \cdot x \geq b \end{array} \qquad\qquad \begin{array}{ll} \max & p^T \cdot b \\ \text{s.t.} & p^T \cdot A = c^T \\ & p > 0 \end{array}$$

i. Formally state the *Complementary Slackness Theorem* using the notation of the above primal-dual pair. **[4 marks]**

ii. Which part of the theorem is trivially fulfilled for the above primal/dual pair? Explain why this is the case? **[2 marks]**

(b) Construct the dual of the following linear program: **[5 marks]**

$$\begin{array}{lrcrcrcrcr} \min & x_1 & - & 3x_2 & + & 2x_3 & & & & \\ s.t. & 2x_1 & + & x_2 & - & 2x_3 & + & 2x_4 & \geq & 2 \\ & -2x_1 & + & 3x_2 & - & x_3 & & & = & 5 \\ & 5x_1 & - & 7x_2 & + & x_3 & + & 3x_4 & \leq & 8 \\ & & & & & & x_1, & x_3 & \geq & 0 \\ & & & & & & & x_2 & \leq & 0 \end{array}$$

(c) Consider the following linear program:

$$\begin{array}{lrcrcrcrcr} \min & 2x_1 & + & 4x_2 & + & 30x_3 & + & x_4 & & \\ s.t. & x_1 & + & x_2 & + & 5x_3 & & & = & 5 \\ & x_1 & & & + & 5x_3 & - & 2x_4 & \leq & 0 \\ & 2x_1 & + & x_2 & + & 4x_3 & + & x_4 & = & 14 \\ & & & & & x_3 & + & x_4 & = & 5 \\ & & & 2x_2 & + & 3x_3 & + & x_4 & \geq & 6 \\ & & & x_1, & & x_2, & x_3, & x_4 & \geq & 0 \end{array}$$

Verify that $x^* = (4, 1, 0, 5)^T$ is optimal, using complementary slackness. **[8 marks]**

(d) Let A be a symmetric square matrix, i.e., $A = A^T$. Consider the linear programming problem

$$\begin{array}{ll} \min & c^T \cdot x \\ \text{s.t.} & A \cdot x \geq c \\ & x \geq 0 \end{array}$$

You are given a basic feasible solution x^*, which satisfies $A \cdot x^* = c$ and $x^* \geq 0$. Show that x^* is an optimal solution. (**Hint:** Construct the dual and use the weak-duality theorem.) **[6 marks]**

4. **Optimisation in Finance**

(a) A company will face the following cash requirements (in thousands of £) in the next four quarters (positive entries represent cash needs while negative entries represent cash surpluses):

Q_1	Q_2	Q_3	Q_4
200	300	−200	−400

(Note: 1 quarter = 3 months.)
Initially, the company has no cash but the following two borrowing possibilities:

- A one year loan available at the beginning of Q_1 with a total interest rate of 5% for the year.
- A quarterly loan (available at the beginning of each quarter) with an interest rate of 2% for the quarter.

Any excess funds can be invested at an interest rate of 0.5% per quarter.

Formulate a linear program that maximises the wealth of the company at the beginning of Q_5. Make sure to explain the meaning of the decision variables. **[10 marks]**

(b) In the lectures we discussed *arbitrage*. Provide a definition for both types (A and B) of arbitrage. **[5 marks]**

(c) Suppose we have the following European Call options, all w.r.t. the same underlying asset (and maturity) which is currently priced at 50 £:

Option i	strike price K_i (in £)	price S_0^i (in £)
1	40	10
2	50	7
3	60	3
4	70	1

Formulate a linear program (LP) that can be used to detect if the above options provide a type-A arbitrage opportunity. Explain how type-A arbitrage can be detected from the solution of your LP. In your LP use the variable x_i for the amount of options i that we buy ($x_i \geq 0$) or sell ($x_i < 0$). **[10 marks]**