Linear Algebra Tutorial

A matrix has rows and columns. Examples: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \rightarrow$ Dimension: 2 x 2

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \rightarrow$ Dimension: 2 x 3

- Transpose of a matrix: $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

A (column) vector of dimension n is an ordered collection of n elements, which are called components.

Example: $x = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} \rightarrow$ Vector of dimension 3.

The transpose of x will be a row vector: $x^T = [3 \quad 2 \quad 2]$.

- Multiplying a vector by a scalar

Example: Given $\vec{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\lambda = 3, \lambda \cdot \vec{a} = 3 \cdot \vec{a} = 3 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$

$$-1 \cdot \vec{a} = -1 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

- Multiplication of matrices

Examples: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{x} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 \cdot 5 + 2 \cdot 7 & 1 \cdot 6 + 2 \cdot 8 \\ 3 \cdot 5 + 4 \cdot 7 & 3 \cdot 6 + 4 \cdot 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$

(2 x 2) (2 x 2) (2 x 2)

$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} \text{x} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 2 \\ 3 & 0 & 0 & 1 \end{bmatrix} =$

(2 x 3) (3 x 4)

$$\begin{bmatrix} 1\cdot1+2\cdot2+1\cdot3 & 1\cdot0+2\cdot1+1\cdot0 & 1\cdot1+2\cdot0+1\cdot0 & 1\cdot0+2\cdot2+1\cdot1 \\ 2\cdot1+1\cdot2+2\cdot3 & 2\cdot0+1\cdot1+2\cdot0 & 2\cdot1+1\cdot0+2\cdot0 & 2\cdot0+1\cdot2+2\cdot1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 2 & 1 & 5 \\ 10 & 1 & 2 & 4 \end{bmatrix}$$

(2 x 4)

$$x \cdot x^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1\cdot1 & 1\cdot2 & 1\cdot3 \\ 2\cdot1 & 2\cdot2 & 2\cdot3 \\ 3\cdot1 & 3\cdot2 & 3\cdot3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

(3 x 1) (1 x 3) (3 x 3)

- Inner Product

$$x^T \cdot x = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1\cdot1+2\cdot2+3\cdot3 = 14$$

(1 x 3) (3 x 1)

$x^T \cdot x = \sum_{i=1}^{n} x_i^T \cdot x_i$

- Determinant of a matrix

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\cdot a_{22} - a_{21}\cdot a_{12}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\cdot\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\cdot\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\cdot\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Given $c \in \Re^2$ with $c = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $b, x \in \Re^2$ and $A \in \Re^{2x2}$ with $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$,

minimise $c^T \cdot x$

subject to $A \cdot x = b$

$$minimise \begin{bmatrix} 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1\cdot x_1 + 2\cdot x_2$$

$$subject\ to \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} \cdot x_1 + a_{12} \cdot x_2 \\ a_{21} \cdot x_1 + a_{22} \cdot x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Given $x \in \Re^3$, $c \in \Re^3$ with $c = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, $b \in \Re^3$ with $b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $A \in \Re^{3x3}$ with $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 3 & 1 \end{bmatrix}$,

maximise $c^T \cdot x$

subject to $A_i \cdot x \leq b_i$, for $i \in M_1 \cup M_2$

$A_i \cdot x \geq b_i$, for $i \in M_3$

where M_i is the i^{th} row of $A \cdot x$

$$maximise\ x_1 + 3 \cdot x_3$$

$$subject\ to\ x_1 + x_3 \leq 1$$

$$2 \cdot x_1 \leq 2$$

$$3 \cdot x_2 + x_3 \geq 3$$