COMP331/557

Chapter 4:
Duality Theory

(Bertsimas & Tsitsiklis, Chapter 4)

Example

$$\begin{array}{rrcrcr}
\text{minimize} & x_1 & + & 2x_2 & & \\
\text{s.t.} & x_1 & & & \geq & 2 \\
& & & x_2 & \geq & 2 \\
& -x_1 & + & x_2 & \geq & 1 \\
& x_1 & + & x_2 & \geq & 5 \\
& & & x_1, x_2 & \geq & 0
\end{array}$$

Goal: Find an upper bound on the optimal solution value z^*.

Easy: Any feasible solution provides one.

Examples:

- $(x_1, x_2) = (4, 5) \quad \Rightarrow z^* \leq 14$
- $(x_1, x_2) = (3, 4) \quad \Rightarrow z^* \leq 11$
- $(x_1, x_2) = (2, 4) \quad \Rightarrow z^* \leq 10$
- $(x_1, x_2) = (2, 3) \quad \Rightarrow z^* \leq 8$

Example

$$
\begin{array}{rrcrcll}
\text{minimize} & x_1 & + & 2x_2 & & & \leftarrow z \\
\text{s.t.} & x_1 & & & \geq & 2 & \leftarrow C_1 \\
 & & & x_2 & \geq & 2 & \leftarrow C_2 \\
 & -x_1 & + & x_2 & \geq & 1 & \leftarrow C_3 \\
 & x_1 & + & x_2 & \geq & 5 & \leftarrow C_4 \\
 & & & x_1, x_2 & \geq & 0 &
\end{array}
$$

New goal: Find a lower bound on the optimal solution value.

Examples:

- $C_4 \quad \Rightarrow \quad z \geq 5$
- $C_1 + 2\ C_2 \quad \Rightarrow \quad z \geq 6$
- $3\ C_1 + 2\ C_3 \quad \Rightarrow$
- $3\ C_2 - C_3 \quad \Rightarrow$

Example

$$\begin{array}{rrcrcrl}
\text{minimize} & x_1 & + & 2x_2 & & & \leftarrow z \\
\text{s.t.} & x_1 & & & \geq & 2 & \leftarrow C_1 \\
& & & x_2 & \geq & 2 & \leftarrow C_2 \\
& -x_1 & + & x_2 & \geq & 1 & \leftarrow C_3 \\
& x_1 & + & x_2 & \geq & 5 & \leftarrow C_4 \\
& & & x_1, x_2 & \geq & 0 &
\end{array}$$

Idea: Add non-negative combination $p_1 \cdot C_1 + p_2 \cdot C_2 + p_3 \cdot C_3 + p_4 \cdot C_4$ of the constraints, s.t.:

$$\begin{aligned}
z = x_1 + 2x_2 &\geq (p_1 - p_3 + p_4) \cdot x_1 + (p_2 + p_3 + p_4) \cdot x_2 \\
&\geq 2\, p_1 + 2\, p_2 + p_3 + 5\, p_4
\end{aligned}$$

Dual Problem:

Find the best such lower bound.

More general

$$\begin{array}{rccccccll}
\text{minimize} & c_1x_1 & + & \cdots & + & c_nx_n & & & \leftarrow z \\
\text{s.t.} & a_{11}x_1 & + & \cdots & + & a_{1n}x_n & \geq & b_1 & \leftarrow C_1 \\
& a_{21}x_1 & + & \cdots & + & a_{2n}x_n & \geq & b_2 & \leftarrow C_2 \\
& \vdots & & \ddots & & & \vdots & & \\
& a_{m1}x_1 & + & \cdots & + & a_{mn}x_n & \geq & b_m & \leftarrow C_m \\
& & & & & x_1, \ldots, x_n & \geq & 0 &
\end{array}$$

Consider: $p_1 C_1 + p_2 C_2 + \cdots + p_m C_m$

Q: What are the conditions on $p_1, \ldots, p_m$ so that this combination lower bounds z?

$$\begin{array}{rcl}
a_{11}p_1 + a_{21}p_2 + \cdots + a_{m1}p_m & \leq & c_1 \\
\vdots \qquad \vdots \qquad \vdots & \leq & \vdots \\
a_{1n}p_1 + a_{1n}p_2 + \cdots + a_{mn}p_m & \leq & c_n \\
p_1, p_2, \ldots, p_m & \geq & 0
\end{array}$$

Q: What lower bound do we get?

Primal and Dual LP

Primal: Decision variables $x_1, \ldots, x_n$.

$$\begin{array}{rcccccccl} \text{minimize} & c_1 x_1 & + & \cdots & + & c_n x_n & & \\ \text{s.t.} & a_{11} x_1 & + & \cdots & + & a_{1n} x_n & \geq & b_1 \\ & a_{21} x_1 & + & \cdots & + & a_{2n} x_n & \geq & b_2 \\ & \vdots & & \ddots & & & \vdots & \\ & a_{m1} x_1 & + & \cdots & + & a_{mn} x_n & \geq & b_m \\ & & & & & x_1, \ldots, x_n & \geq & 0 \end{array}$$

$$\begin{array}{lll} \min & c^T x & \\ \text{s.t.} & A x & \geq b \\ & x & \geq 0 \end{array}$$

Dual: Decision variables $p_1, \ldots, p_m$.

$$\begin{array}{rcccccccl} \text{maximize} & b_1 p_1 & + & \cdots & + & b_m p_m & & \\ \text{s.t.} & a_{11} p_1 & + & \cdots & + & a_{m1} p_m & \leq & c_1 \\ & a_{12} p_1 & + & \cdots & + & a_{m2} p_m & \leq & c_2 \\ & \vdots & & \ddots & & & \vdots & \\ & a_{1n} p_1 & + & \cdots & + & a_{mn} p_m & \leq & c_n \\ & & & & & p_1, \ldots, p_m & \geq & 0 \end{array}$$

$$\begin{array}{lll} \max & b^T p & \\ \text{s.t.} & A^T p & \leq c \\ & p & \geq 0 \end{array}$$

Primal and Dual Example (1)

Primal:

$$
\begin{array}{lrcrcl}
\min & x_1 & + & 2x_2 & & \\
\text{s.t.} & 2x_1 & + & x_2 & \geq & 7 \\
 & -x_1 & + & 3x_2 & \geq & 1 \\
 & x_1 & + & 4x_2 & \geq & 5 \\
 & & & x_1, x_2 & \geq & 0
\end{array}
$$

Dual:

Primal and Dual Example (2)

Primal:

$$\begin{array}{lrcrcr} \min & -x_1 & + & 4x_2 & & \\ \text{s.t.} & 3x_1 & + & 2x_2 & \geq & 9 \\ & x_1 & - & 3x_2 & \leq & 3 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

Dual:

Primal and Dual Example (3)

Primal:

$$\begin{array}{rrcrcr} \min & -x_1 & + & 4x_2 & & \\ \text{s.t.} & 3x_1 & + & 2x_2 & \geq & 9 \\ & x_1 & - & 3x_2 & = & 3 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

Dual:

Primal and Dual Linear Program

Consider the general linear program:

$$
\begin{array}{lll}
\min & c^T \cdot x & \\
\text{s.t.} & {a_i}^T \cdot x \geq b_i & \text{for } i \in M_1 \\
 & {a_i}^T \cdot x \leq b_i & \text{for } i \in M_2 \\
 & {a_i}^T \cdot x = b_i & \text{for } i \in M_3 \\
 & x_j \geq 0 & \text{for } j \in N_1 \\
 & x_j \leq 0 & \text{for } j \in N_2 \\
 & x_j \text{ free} & \text{for } j \in N_3
\end{array}
$$

Obtain a lower bound:

$$
\begin{array}{lll}
\max & p^T \cdot b & \\
\text{s.t.} & p_i \geq 0 & \text{for } i \in M_1 \\
 & p_i \leq 0 & \text{for } i \in M_2 \\
 & p_i \text{ free} & \text{for } i \in M_3 \\
 & A_j^T \cdot p \leq c_j & \text{for } j \in N_1 \\
 & A_j^T \cdot p \geq c_j & \text{for } j \in N_2 \\
 & A_j^T \cdot p = c_j & \text{for } j \in N_3
\end{array}
$$

The linear program on the right hand side is the dual linear program of the primal linear program on the left hand side.

Primal and Dual Variables and Constraints

primal LP (minimize)		dual LP (maximize)	
	$\geq b_i$	≥ 0	
constraints	$\leq b_i$	≤ 0	variables
	$= b_i$	free	
	≥ 0	$\leq c_i$	
variables	≤ 0	$\geq c_i$	constraints
	free	$= c_i$	

Examples

primal LP	dual LP
$\begin{array}{lrl} \min & c^T \cdot x & \\ \text{s.t.} & A \cdot x & \geq b \end{array}$	$\begin{array}{lrl} \max & p^T \cdot b & \\ \text{s.t.} & A^T \cdot p & = c \\ & p & \geq 0 \end{array}$
$\begin{array}{lrl} \min & c^T \cdot x & \\ \text{s.t.} & A \cdot x & = b \\ & x & \geq 0 \end{array}$	$\begin{array}{lrl} \max & p^T \cdot b & \\ \text{s.t.} & A^T \cdot p & \leq c \end{array}$

Basic Properties of the Dual Linear Program

Theorem 4.1.

The dual of the dual LP is the primal LP.

Proof:

Primal in general form:

$$
\begin{array}{lll}
\min & c^T \cdot x & \\
\text{s.t.} & a_i{}^T \cdot x \geq b_i & \text{for } i \in M_1 \\
& a_i{}^T \cdot x \leq b_i & \text{for } i \in M_2 \\
& a_i{}^T \cdot x = b_i & \text{for } i \in M_3 \\
& x_j \geq 0 & \text{for } j \in N_1 \\
& x_j \leq 0 & \text{for } j \in N_2 \\
& x_j \text{ free} & \text{for } j \in N_3
\end{array}
$$

Dual:

$$
\begin{array}{lll}
\max & p^T \cdot b & \\
\text{s.t.} & p_i \geq 0 & \text{for } i \in M_1 \\
& p_i \leq 0 & \text{for } i \in M_2 \\
& p_i \text{ free} & \text{for } i \in M_3 \\
& A_j^T \cdot p \leq c_j & \text{for } j \in N_1 \\
& A_j^T \cdot p \geq c_j & \text{for } j \in N_2 \\
& A_j^T \cdot p = c_j & \text{for } j \in N_3
\end{array}
$$

Basic Properties of the Dual Linear Program

Proof (cont.):

Dual:

$$
\begin{array}{lll}
\max & p^T \cdot b & \\
\text{s.t.} & p_i \geq 0 & \text{for } i \in M_1 \\
& p_i \leq 0 & \text{for } i \in M_2 \\
& p_i \text{ free} & \text{for } i \in M_3 \\
& A_j^T \cdot p \leq c_j & \text{for } j \in N_1 \\
& A_j^T \cdot p \geq c_j & \text{for } j \in N_2 \\
& A_j^T \cdot p = c_j & \text{for } j \in N_3
\end{array}
$$

Dual (in primal form):

$$
\begin{array}{lll}
\min & -p^T \cdot b & \\
\text{s.t.} & -A_j^T \cdot p \geq -c_j & \text{for } j \in N_1 \\
& -A_j^T \cdot p \leq -c_j & \text{for } j \in N_2 \\
& -A_j^T \cdot p = -c_j & \text{for } j \in N_3 \\
& p_i \geq 0 & \text{for } i \in M_1 \\
& p_i \leq 0 & \text{for } i \in M_2 \\
& p_i \text{ free} & \text{for } i \in M_3
\end{array}
$$

Basic Properties of the Dual Linear Program

Proof (cont.):

Dual (in primal form):

$$
\begin{aligned}
\min \quad & -p^T \cdot b & \\
\text{s.t.} \quad & -A_j^T \cdot p \geq -c_j & \text{for } j \in N_1 \\
& -A_j^T \cdot p \leq -c_j & \text{for } j \in N_2 \\
& -A_j^T \cdot p = -c_j & \text{for } j \in N_3 \\
& p_i \geq 0 & \text{for } i \in M_1 \\
& p_i \leq 0 & \text{for } i \in M_2 \\
& p_i \text{ free} & \text{for } i \in M_3
\end{aligned}
$$

Dual of Dual:

Equavalence of the Dual LP

Theorem 4.2.

Let Π_1 and Π_2 be two LPs where Π_2 has been obtained from Π_1 by (several) transformations of the following type:

i. replace a free variable by the difference of two non-negative variables;
ii. introduce a slack variable in order to replace an inequality constraint by an equation;
iii. if some row of a feasible equality system is a linear combination of the other rows, eliminate this row.

Then the dual of Π_1 is equivalent to the dual of Π_2.

Weak Duality Theorem

Theorem 4.3.

If x is a feasible solution to the primal LP (minimization problem) and p a feasible solution to the dual LP (maximization problem), then

$$c^T \cdot x \geq p^T \cdot b \quad .$$

Corollary 4.4.

Consider a primal-dual pair of linear programs as above.

a. If the primal LP is unbounded (i. e., optimal cost $= -\infty$), then the dual LP is infeasible.
b. If the dual LP is unbounded (i. e., optimal cost $= \infty$), then the primal LP is infeasible.
c. If x and p are feasible solutions to the primal and dual LP, resp., and if $c^T \cdot x = p^T \cdot b$, then x and p are optimal solutions.

Strong Duality Theorem

Theorem 4.5.

If an LP has an optimal solution, so does its dual and the optimal costs are equal.

Different Possibilities for Primal and Dual LP

primal \ dual	finite optimum	unbounded	infeasible
finite optimum	possible	impossible	impossible
unbounded	impossible	impossible	possible
infeasible	impossible	possible	possible

Example of infeasible primal and dual LP:

$$
\begin{aligned}
\min \quad & x_1 + 2\,x_2 \\
\text{s.t.} \quad & x_1 + x_2 = 1 \\
& 2\,x_1 + 2\,x_2 = 3
\end{aligned}
\qquad\qquad
\begin{aligned}
\max \quad & p_1 + 3\,p_2 \\
\text{s.t.} \quad & p_1 + 2\,p_2 = 1 \\
& p_1 + 2\,p_2 = 2
\end{aligned}
$$

Complementary Slackness

Consider the following pair of primal and dual LPs:

$$\begin{array}{ll} \min & c^T \cdot x \\ \text{s.t.} & A \cdot x \geq b \end{array} \qquad\qquad \begin{array}{ll} \max & p^T \cdot b \\ \text{s.t.} & p^T \cdot A = c^T \\ & p \geq 0 \end{array}$$

If x and p are feasible solutions, then $c^T \cdot x = p^T \cdot A \cdot x \geq p^T \cdot b$.Thus,

$$c^T \cdot x = p^T \cdot b \quad \Longleftrightarrow \quad \text{for all } i\text{: } p_i = 0 \text{ if } {a_i}^T \cdot x > b_i.$$

Theorem 4.6.

Consider an arbitrary pair of primal and dual LPs. Let x and p be feasible solutions to the primal and dual LP, respectively. Then x and p are both optimal if and only if

$$u_i := p_i\,({a_i}^T \cdot x - b_i) = 0 \quad \text{for all } i, \tag{1}$$

$$v_j := (c_j - p^T \cdot A_j)\,x_j = 0 \quad \text{for all } j. \tag{2}$$

Complementary Slackness Example

Consider the following LP in standard form and its dual:

$$\begin{aligned} \min \quad & 13x_1 + 10x_2 + 6x_3 \\ \text{s.t.} \quad & 5x_1 + x_2 + 3x_3 = 8 \\ & 3x_1 + x_2 = 3 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

$$\begin{aligned} \max \quad & 8p_1 + 3p_2 \\ \text{s.t.} \quad & 5p_1 + 3p_2 \leq 13 \\ & p_1 + p_2 \leq 10 \\ & 3p_1 \leq 6 \end{aligned}$$

Claim: $x^* = (1, 0, 1)$ is a non-degenerate optimal solution to the primal.
Verify this using complementary slackness!

Geometric View

Consider pair of primal and dual LPs with $A \in \mathbb{R}^{m \times n}$ and $\text{rank}(A) = n$:

$$\begin{aligned} \min \quad & c^T \cdot x \\ \text{s.t.} \quad & {a_i}^T \cdot x \geq b_i, \quad i = 1, \ldots, m \end{aligned} \qquad\qquad \begin{aligned} \max \quad & p^T \cdot b \\ \text{s.t.} \quad & \sum_{i=1}^{m} p_i \cdot a_i = c \\ & p \geq 0 \end{aligned}$$

Let $I \subseteq \{1, \ldots, m\}$ with $|I| = n$ and a_i, $i \in I$, linearly independent.

$\Longrightarrow \quad {a_i}^T \cdot x = b_i$, $i \in I$, has unique solution x^I (basic solution)

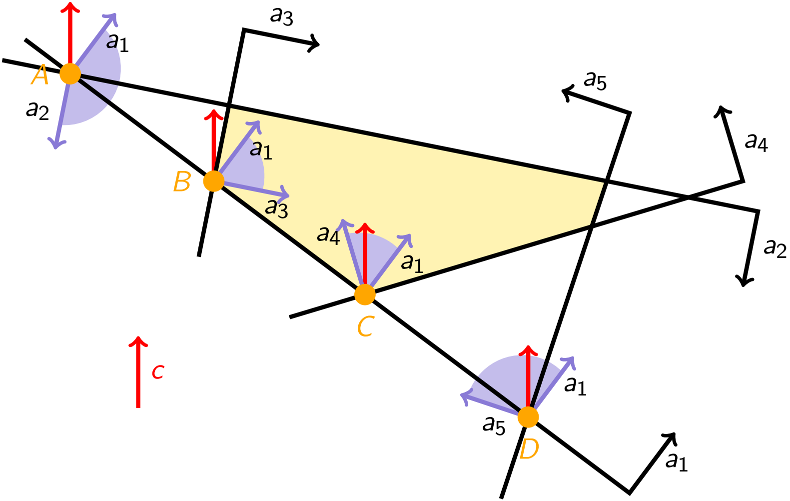
Let $p \in \mathbb{R}^m$ (dual vector). Then x, p are optimal solutions if

- i ${a_i}^T \cdot x \geq b_i$ for all i (primal feasibility)
- ii $p_i = 0$ for all $i \notin I$ (complementary slackness)
- iii $\sum_{i=1}^{m} p_i \cdot a_i = c$ (dual feasibility)
- iv $p \geq 0$ (dual feasibility)

(ii) and (iii) imply $\sum_{i \in I} p_i \cdot a_i = c$ which has a unique solution p^I.

The a_i, $i \in I$, form basis for dual LP and p^I is corresponding basic solution.

Geometric View (cont.)



Dual Variables as Marginal Costs

Consider the primal dual pair:

$$\begin{aligned} \min \quad & c^T \cdot x \\ \text{s.t.} \quad & A \cdot x = b \\ & x \geq 0 \end{aligned} \qquad\qquad \begin{aligned} \max \quad & p^T \cdot b \\ \text{s.t.} \quad & p^T \cdot A \leq c^T \end{aligned}$$

Let x^* be optimal basic feasible solution to primal LP with basis B, i. e., $x_B^* = B^{-1} \cdot b$ and assume that $x_B^* > 0$ (i. e., x^* non-degenerate).

Replace b by $b + d$. For small d, the basis B remains feasible and optimal:

$$B^{-1} \cdot (b + d) = B^{-1} \cdot b + B^{-1} \cdot d \geq 0 \qquad \text{(feasibility)}$$

$$\bar{c}^T = c^T - {c_B}^T \cdot B^{-1} \cdot A \geq 0 \qquad \text{(optimality)}$$

Optimal cost of perturbed problem is

$${c_B}^T \cdot B^{-1} \cdot (b + d) = {c_B}^T \cdot x_B^* + \underbrace{({c_B}^T \cdot B^{-1})}_{= p^T} \cdot d$$

Thus, p_i is the marginal cost per unit increase of b_i.

Dual Variables as Shadow Prices

Diet problem:

- $a_{ij} :=$ amount of nutrient i in one unit of food j
- $b_i :=$ requirement of nutrient i in some ideal diet
- $c_j :=$ cost of one unit of food j on the food market

LP duality: Let $x_j :=$ number of units of food j in the diet:

$$\begin{array}{lll} \min & c^T \cdot x \\ \text{s.t.} & A \cdot x = b \\ & x \geq 0 \end{array} \qquad\qquad \begin{array}{ll} \max & p^T \cdot b \\ \text{s.t.} & p^T \cdot A \leq c^T \end{array}$$

Dual interpretation:

- p_i is “fair” price per unit of nutrient i
- $p^T \cdot A_j$ is value of one unit of food j on the nutrient market
- food j used in ideal diet ($x_j^* > 0$) is consistently priced at the two markets (by complementary slackness)
- ideal diet has the same value on both markets (by strong duality)

Dual Basic Solutions

Consider LP in standard form with $A \in \mathbb{R}^{m \times n}$, $\text{rank}(A) = m$, and dual LP:

$$\begin{aligned} \min \quad & c^T \cdot x \\ \text{s.t.} \quad & A \cdot x = b \\ & x \geq 0 \end{aligned} \qquad\qquad \begin{aligned} \max \quad & p^T \cdot b \\ \text{s.t.} \quad & p^T \cdot A \leq c^T \end{aligned}$$

Observation 4.7.

A basis B yields

- a primal basic solution given by $x_B := B^{-1} \cdot b$ and
- a dual basic solution $p^T := c_B{}^T \cdot B^{-1}$.

Moreover,

a. the values of the primal and the dual basic solutions are equal:
$$c_B{}^T \cdot x_B = c_B{}^T \cdot B^{-1} \cdot b = p^T \cdot b \;\; ;$$
b. p is feasible if and only if $\bar{c} \geq 0$;
c. reduced cost $\bar{c}_i = 0$ corresponds to active dual constraint;
d. p is degenerate if and only if $\bar{c}_i = 0$ for some non-basic variable x_i.

Dual Simplex Method

- Let B be a basis whose corresponding dual basic solution p is feasible.
- If also the primal basic solution x is feasible, then x, p are optimal.
- Assume that $x_{B(\ell)} < 0$ and consider the ℓth row of the simplex tableau

$$(x_{B(\ell)}, v_1, \dots, v_n) \qquad \text{(pivot row)}$$

I Let $j \in \{1, \dots, n\}$ with $v_j < 0$ and

$$\frac{\bar{c}_j}{|v_j|} = \min_{i: v_i < 0} \frac{\bar{c}_i}{|v_i|}$$

Performing an iteration of the simplex method with pivot element v_j yields new basis B' and corresponding dual basic solution p' with

$$c_{B'}{}^T \cdot B'^{-1} \cdot A \leq c^T \quad \text{and} \quad p'^T \cdot b \geq p^T \cdot b \quad (\text{with} > \text{if } \bar{c}_j > 0).$$

II If $v_i \geq 0$ for all $i \in \{1, \dots, n\}$, then the dual LP is unbounded and the primal LP is infeasible.

Dual Simplex Example

		x_1	x_2	x_3	x_4	x_5
	0	2	6	10	0	0
$x_4 =$	2	-2	4	1	1	0
$x_5 =$	-1	4	-2	-3	0	1

- Determine pivot row ($x_5 < 0$)

Dual Simplex Example

		x_1	x_2	x_3	x_4	x_5
	0	2	6	10	0	0
$x_4 =$	2	-2	4	1	1	0
$x_5 =$	-1	4	-2	-3	0	1

- Determine pivot row ($x_5 < 0$)
- Find pivot column.
  - Column 2 and 3 have negative entries in pivot row.

Dual Simplex Example

		x_1	x_2	x_3	x_4	x_5
	0	2	6	10	0	0
$x_4 =$	2	-2	4	1	1	0
$x_5 =$	-1	4	-2	-3	0	1

- Determine pivot row ($x_5 < 0$)
- Find pivot column.
  - Column 2 and 3 have negative entries in pivot row.
  - Column 2 attains minimum.

Dual Simplex Example

		x_1	x_2	x_3	x_4	x_5
	0	2	6	10	0	0
$x_4 =$	2	-2	4	1	1	0
$x_5 =$	-1	4	-2	-3	0	1

- Determine pivot row ($x_5 < 0$)
- Find pivot column.
  - Column 2 and 3 have negative entries in pivot row.
  - Column 2 attains minimum.
- Perform basis change:
  - x_5 leaves and x_2 enters basis.
  - Eliminate other entries in the pivot column.
  - Divide pivot row by pivot element.

Dual Simplex Example

		x_1	x_2	x_3	x_4	x_5
	-3	14	0	1	0	3
$x_4 =$	2	-2	4	1	1	0
$x_5 =$	-1	4	-2	-3	0	1

- Determine pivot row ($x_5 < 0$)
- Find pivot column.
  - Column 2 and 3 have negative entries in pivot row.
  - Column 2 attains minimum.
- Perform basis change:
  - x_5 leaves and x_2 enters basis.
  - Eliminate other entries in the pivot column.
  - Divide pivot row by pivot element.

Dual Simplex Example

		x_1	x_2	x_3	x_4	x_5
	-3	14	0	1	0	3
$x_4 =$	0	6	0	-5	1	2
$x_5 =$	-1	4	-2	-3	0	1

- Determine pivot row ($x_5 < 0$)
- Find pivot column.
  - Column 2 and 3 have negative entries in pivot row.
  - Column 2 attains minimum.
- Perform basis change:
  - x_5 leaves and x_2 enters basis.
  - Eliminate other entries in the pivot column.
  - Divide pivot row by pivot element.

Dual Simplex Example

		x_1	x_2	x_3	x_4	x_5
	-3	14	0	1	0	3
$x_4 =$	0	6	0	-5	1	2
$x_2 =$	$1/2$	-2	1	$3/2$	0	$-1/2$

- Determine pivot row ($x_5 < 0$)
- Find pivot column.
  - Column 2 and 3 have negative entries in pivot row.
  - Column 2 attains minimum.
- Perform basis change:
  - x_5 leaves and x_2 enters basis.
  - Eliminate other entries in the pivot column.
  - Divide pivot row by pivot element.

Remarks on the Dual Simplex Method

- Dual simplex method terminates if lexicographic pivoting rule is used:
  - Choose any row ℓ with $x_{B(\ell)} < 0$ to be the pivot row.
  - Among all columns j with $v_j < 0$ choose the one which is lexicographically minimal when divided by $|v_j|$.
- Dual simplex method is useful if, e. g., dual basic solution is readily available.
- Example: Resolve LP after right-hand-side b has changed.