Chapter 2: Linear Programming Basics

(Bertsimas & Tsitsiklis, Chapter 1)

Example of a Linear Program

$$
\begin{array}{lrcrcrcrl}
\text{minimize} & 2x_1 & - & x_2 & + & 4x_3 & & & \\
\text{subject to} & x_1 & + & x_2 & & & + & x_4 & \leq 2 \\
 & & & 3x_2 & - & x_3 & & & = 5 \\
 & & & & & x_3 & + & x_4 & \geq 3 \\
 & x_1 & & & & & & & \geq 0 \\
 & & & & & x_3 & & & \leq 0
\end{array}
$$

Remarks.

- objective function is linear in vector of variables $x = (x_1, x_2, x_3, x_4)^T$
- constraints are linear inequalities and linear equations
- last two constraints are special
  (non-negativity and non-positivity constraint, respectively)

General Linear Program

$$
\begin{aligned}
\text{minimize} \quad & c^T \cdot x \\
\text{subject to} \quad & {a_i}^T \cdot x \geq b_i && \text{for } i \in M_1, && (2.1)\\
& {a_i}^T \cdot x = b_i && \text{for } i \in M_2, && (2.2)\\
& {a_i}^T \cdot x \leq b_i && \text{for } i \in M_3, && (2.3)\\
& x_j \geq 0 && \text{for } j \in N_1, && (2.4)\\
& x_j \leq 0 && \text{for } j \in N_2, && (2.5)
\end{aligned}
$$

with $c \in \mathbb{R}^n$, $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$ for $i \in M_1 \dot\cup M_2 \dot\cup M_3$ (finite index sets), and $N_1, N_2 \subseteq \{1, \ldots, n\}$ given.

- $x \in \mathbb{R}^n$ satisfying constraints (2.1) – (2.5) is a feasible solution.
- feasible solution x^* is optimal solution if

$$c^T \cdot x^* \leq c^T \cdot x \qquad \text{for all feasible solutions } x.$$

- linear program is unbounded if, for all $k \in \mathbb{R}$, there is a feasible solution $x \in \mathbb{R}^n$ with $c^T \cdot x \leq k$.

Special Forms of Linear Programs

- maximizing $c^T \cdot x$ is equivalent to minimizing $(-c)^T \cdot x$.
- any linear program can be written in the form

$$\begin{aligned} \text{minimize} \quad & c^T \cdot x \\ \text{subject to} \quad & A \cdot x \geq b \end{aligned}$$

  for some $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$:

  - rewrite $a_i{}^T \cdot x = b_i$ as: $a_i{}^T \cdot x \geq b_i \quad \wedge \quad a_i{}^T \cdot x \leq b_i$,
  - rewrite $a_i{}^T \cdot x \leq b_i$ as: $(-a_i)^T \cdot x \geq -b_i$.

- Linear program in standard form:

$$\begin{aligned} \min \quad & c^T \cdot x \\ \text{s.t.} \quad & A \cdot x = b \\ & x \geq 0 \end{aligned}$$

  with $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$.

Example: Diet Problem

Given:

- n different foods, m different nutrients
- $a_{ij} :=$ amount of nutrient i in one unit of food j
- $b_i :=$ requirement of nutrient i in some ideal diet
- $c_j :=$ cost of one unit of food j

Task: find a cheapest ideal diet consisting of foods $1, \ldots, n$.

LP formulation: Let $x_j :=$ number of units of food j in the diet:

$$\begin{aligned} \min \quad & c^T \cdot x \\ \text{s.t.} \quad & A \cdot x = b \\ & x \geq 0 \end{aligned} \qquad \text{or} \qquad \begin{aligned} \min \quad & c^T \cdot x \\ \text{s.t.} \quad & A \cdot x \geq b \\ & x \geq 0 \end{aligned}$$

with $A = (a_{ij}) \in \mathbb{R}^{m \times n}$, $b = (b_i) \in \mathbb{R}^m$, $c = (c_j) \in \mathbb{R}^n$.

Reduction to Standard Form

Any linear program can be brought into standard form:

- elimination of free (unbounded) variables x_j:

  replace x_j with $x_j^+, x_j^- \geq 0$: $\quad x_j = x_j^+ - x_j^-$

- elimination of non-positive variables x_j:

  replace $x_j \leq 0$ with $(-x_j) \geq 0$.

- elimination of inequality constraint ${a_i}^T \cdot x \leq b_i$:

  introduce slack variable $s \geq 0$ and rewrite: ${a_i}^T \cdot x + s = b_i$

- elimination of inequality constraint ${a_i}^T \cdot x \geq b_i$:

  introduce slack variable $s \geq 0$ and rewrite: ${a_i}^T \cdot x - s = b_i$

Example

The linear program

$$\begin{array}{lrcrl} \min & 2\,x_1 & + & 4\,x_2 & \\ \text{s.t.} & x_1 & + & x_2 & \geq 3 \\ & 3\,x_1 & + & 2\,x_2 & = 14 \\ & x_1 & & & \geq 0 \end{array}$$

is equivalent to the standard form problem

$$\begin{array}{lrcrcrcrl} \min & 2\,x_1 & + & 4\,x_2^+ & - & 4\,x_2^- & & & \\ \text{s.t.} & x_1 & + & x_2^+ & - & x_2^- & - & x_3 & = 3 \\ & 3\,x_1 & + & 2\,x_2^+ & - & 2\,x_2^- & & & = 14 \\ & & & & & & x_1,\, x_2^+,\, x_2^-,\!\!\! & x_3 & \geq 0 \end{array}$$

Affine Linear and Convex Functions

Lemma 2.1.

- **a** An affine linear function $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(x) = c^T \cdot x + d$ with $c \in \mathbb{R}^n$, $d \in \mathbb{R}$, is both convex and concave.
- **b** If $f_1, \ldots, f_k : \mathbb{R}^n \to \mathbb{R}$ are convex functions, then $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(x) := \max_{i=1,\ldots,k} f_i(x)$ is also convex.

Piecewise Linear Convex Objective Functions

Let $c_1, \ldots, c_k \in \mathbb{R}^n$ and $d_1, \ldots, d_k \in \mathbb{R}$.

Consider piecewise linear convex function: $x \mapsto \max_{i=1,\ldots,k} {c_i}^T \cdot x + d_i$:

$$\begin{array}{ll} \min & \max\limits_{i=1,\ldots,k} {c_i}^T \cdot x + d_i \\ \text{s.t.} & A \cdot x \geq b \end{array} \quad \longleftrightarrow \quad \begin{array}{lll} \min & z & \\ \text{s.t.} & z \geq {c_i}^T \cdot x + d_i & \text{for all } i \\ & A \cdot x \geq b & \end{array}$$

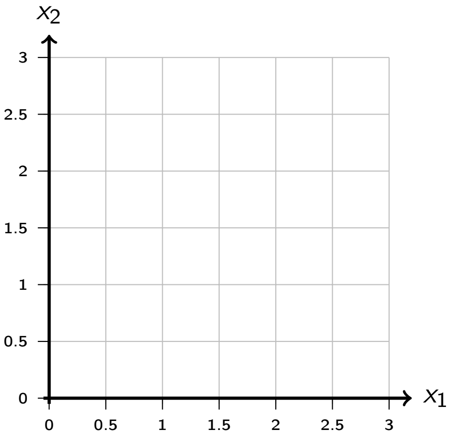
Example: let $c_1, \ldots, c_n \geq 0$

$$\begin{array}{ll} \min & \sum\limits_{i=1}^{n} c_i \cdot |x_i| \\ \text{s.t.} & A \cdot x \geq b \end{array} \quad \leftrightarrow \quad \begin{array}{ll} \min & \sum\limits_{i=1}^{n} c_i \cdot z_i \\ \text{s.t.} & z_i \geq x_i \\ & z_i \geq -x_i \\ & A \cdot x \geq b \end{array}$$

Graphical Representation and Solution

2D example:

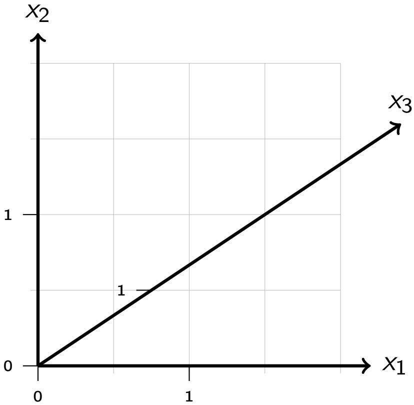
$$
\begin{array}{rrrrl}
\min & -x_1 & - & x_2 & \\
\text{s.t.} & x_1 & + & 2\,x_2 & \leq 3 \\
 & 2\,x_1 & + & x_2 & \leq 3 \\
 & & & x_1,\, x_2 & \geq 0
\end{array}
$$



Graphical Representation and Solution (cont.)

3D example:

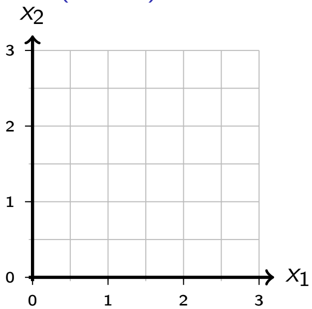
$$
\begin{aligned}
\min \quad & -x_1 - x_2 - x_3 \\
\text{s.t.} \quad & x_1 \leq 1 \\
& x_2 \leq 1 \\
& x_3 \leq 1 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$



Graphical Representation and Solution (cont.)

another 2D example:

$$\begin{array}{lrcrl} \min & c_1\, x_1 & + & c_2\, x_2 & \\ \text{s.t.} & -x_1 & + & x_2 & \leq 1 \\ & & & x_1,\, x_2 & \geq 0 \end{array}$$



- for $c = (1,1)^T$, the unique optimal solution is $x = (0,0)^T$
- for $c = (1,0)^T$, the optimal solutions are exactly the points

$$x = (0, x_2)^T \qquad \text{with } 0 \leq x_2 \leq 1$$

- for $c = (0,1)^T$, the optimal solutions are exactly the points

$$x = (x_1, 0)^T \qquad \text{with } x_1 \geq 0$$

- for $c = (-1,-1)^T$, the problem is unbounded, optimal cost is $-\infty$
- if we add the constraint $x_1 + x_2 \leq -1$, the problem is infeasible

Properties of the Set of Optimal Solutions

In the last example, the following 5 cases occurred:

i there is a unique optimal solution

ii there exist infinitely many optimal solutions, but the set of optimal solutions is bounded

iii there exist infinitely many optimal solutions and the set of optimal solutions is unbounded

iv the problem is unbounded, i. e., the optimal cost is $-\infty$ and no feasible solution is optimal

v the problem is infeasible, i. e., the set of feasible solutions is empty

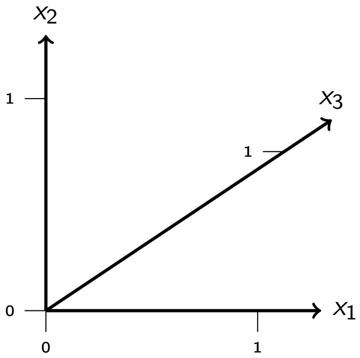
These are indeed all cases that can occur in general (see later).

Visualizing LPs in Standard Form

Example:

Let $A = (1, 1, 1) \in \mathbb{R}^{1\times 3}$, $b = (1) \in \mathbb{R}^1$ and consider the set of feasible solutions

$$P = \{x \in \mathbb{R}^3 \mid A \cdot x = b,\ x \geq 0\} \ .$$



Visualizing LPs in Standard Form

More general:

- if $A \in \mathbb{R}^{m \times n}$ with $m \leq n$ and the rows of A are linearly independent, then

$$\{x \in \mathbb{R}^n \mid A \cdot x = b\}$$

  is an $(n - m)$-dimensional affine subspace in $\mathbb{R}^n$.
- set of feasible solutions lies in this affine subspace and is only constrained by non-negativity constraints $x \geq 0$.