COMP331/557

Chapter 3: The Geometry of Linear Programming

(Bertsimas & Tsitsiklis, Chapter 2)

Polyhedra and Polytopes

Definition 3.1.

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

a. set $\{x \in \mathbb{R}^n \mid A \cdot x \geq b\}$ is called polyhedron
b. $\{x \mid A \cdot x = b,\ x \geq 0\}$ is polyhedron in standard form representation

Definition 3.2.

a. Set $S \subseteq \mathbb{R}^n$ is bounded if there is $K \in \mathbb{R}$ such that

$$\|x\|_\infty \leq K \qquad \text{for all } x \in S.$$

b. A bounded polyhedron is called polytope.

Hyperplanes and Halfspaces

Definition 3.3.

Let $a \in \mathbb{R}^n \setminus \{0\}$ and $b \in \mathbb{R}$:

- a) set $\{x \in \mathbb{R}^n \mid a^T \cdot x = b\}$ is called hyperplane
- b) set $\{x \in \mathbb{R}^n \mid a^T \cdot x \geq b\}$ is called halfspace

Remarks

- Hyperplanes and halfspaces are convex sets.
- A polyhedron is an intersection of finitely many halfspaces.

Convex Combination and Convex Hull

Definition 3.4.

Let $x^1, \ldots, x^k \in \mathbb{R}^n$ and $\lambda_1, \ldots, \lambda_k \in \mathbb{R}_{\geq 0}$ with $\lambda_1 + \cdots + \lambda_k = 1$.

a. The vector $\sum_{i=1}^{k} \lambda_i \cdot x^i$ is a convex combination of $x^1, \ldots, x^k$.
b. The convex hull of $x^1, \ldots, x^k$ is the set of all convex combinations.

Convex Sets, Convex Combinations, and Convex Hulls

Theorem 3.5.

a. The intersection of convex sets is convex.
b. Every polyhedron is a convex set.
c. A convex combination of a finite number of elements of a convex set also belongs to that set.
d. The convex hull of finitely many vectors is a convex set.

Corollary 3.6.

The convex hull of $x^1, \ldots, x^k \in \mathbb{R}^n$ is the smallest (w.r.t. inclusion) convex subset of $\mathbb{R}^n$ containing $x^1, \ldots, x^k$.

Extreme Points and Vertices of Polyhedra

Definition 3.7.

Let $P \subseteq \mathbb{R}^n$ be a polyhedron.

a. $x \in P$ is an extreme point of P if

$$x \neq \lambda \cdot y + (1 - \lambda) \cdot z \qquad \text{for all } y, z \in P \setminus \{x\},\ 0 \leq \lambda \leq 1,$$

i. e., x is not a convex combination of two other points in P.

b. $x \in P$ is a vertex of P if there is some $c \in \mathbb{R}^n$ such that

$$c^T \cdot x < c^T \cdot y \qquad \text{for all } y \in P \setminus \{x\},$$

i. e., x is the unique optimal solution to the LP $\min\{c^T \cdot z \mid z \in P\}$.

Active and Binding Constraints

In the following, let $P \subseteq \mathbb{R}^n$ be a polyhedron defined by

$$
\begin{aligned}
{a_i}^T \cdot x &\geq b_i && \text{for } i \in M_1, \\
{a_i}^T \cdot x &= b_i && \text{for } i \in M_2,
\end{aligned}
$$

with $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$, for all i.

Definition 3.8.

If $x^* \in \mathbb{R}^n$ satisfies ${a_i}^T \cdot x^* = b_i$ for some i, then the corresponding constraint is active (or binding) at x^*.

Basic Facts from Linear Algebra

Theorem 3.9.

Let $x^* \in \mathbb{R}^n$ and $I = \{i \mid {a_i}^T \cdot x^* = b_i\}$. The following are equivalent:

i. there are n vectors in $\{a_i \mid i \in I\}$ which are linearly independent;
ii. the vectors in $\{a_i \mid i \in I\}$ span $\mathbb{R}^n$;
iii. x^* is the unique solution to the system of equations ${a_i}^T \cdot x = b_i$, $i \in I$.

Vertices, Extreme Points, and Basic Feasible Solutions

Definition 3.10.

a $x^* \in \mathbb{R}^n$ is a basic solution of P if
  - all equality constraints are active and
  - there are n linearly independent constraints that are active.

b A basic solution satisfying all constraints is a basic feasible solution.

Theorem 3.11.

For $x^* \in P$, the following are equivalent:

i x^* is a vertex of P;

ii x^* is an extreme point of P;

iii x^* is a basic feasible solution of P.

Number of Vertices

Corollary 3.12.

a. A polyhedron has a finite number of vertices and basic solutions.

b. For a polyhedron in $\mathbb{R}^n$ given by linear equations and m linear inequalities, this number is at most $\binom{m}{n}$.

Example:

$P := \{x \in \mathbb{R}^n \mid 0 \leq x_i \leq 1, \; i = 1, \ldots, n\}$ (n-dimensional unit cube)

- number of constraints: $m = 2n$
- number of vertices: 2^n

Adjacent Basic Solutions and Edges

Definition 3.13.

Let $P \subseteq \mathbb{R}^n$ be a polyhedron.

a. Two distinct basic solutions are adjacent if there are $n - 1$ linearly independent constraints that are active at both of them.
b. If both solutions are feasible, the line segment that joins them is an edge of P.

Polyhedra in Standard Form

Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $P = \{x \in \mathbb{R}^n \mid A \cdot x = b,\ x \geq 0\}$.

Observation

One can assume without loss of generality that $\text{rank}(A) = m$.

Theorem 3.14.

$x \in \mathbb{R}^n$ is a basic solution of P if and only if $A \cdot x = b$ and there are indices $B(1), \dots, B(m) \in \{1, \dots, n\}$ such that

- columns $A_{B(1)}, \dots, A_{B(m)}$ of matrix A are linearly independent and
- $x_i = 0$ for all $i \notin \{B(1), \dots, B(m)\}$.

- $x_{B(1)}, \dots, x_{B(m)}$ are basic variables, the remaining variables non-basic.
- The vector of basic variables is denoted by $x_B := (x_{B(1)}, \dots, x_{B(m)})^T$.
- $A_{B(1)}, \dots, A_{B(m)}$ are basic columns of A and form a basis of $\mathbb{R}^m$.
- The matrix $B := (A_{B(1)}, \dots, A_{B(m)}) \in \mathbb{R}^{m \times m}$ is called basis matrix.

Example:

Consider the following LP:

$$
\begin{array}{lrrrrrrrll}
\min & 2x_1 & & & +x_4 & & & +5x_7 & & \\
\text{s.t.} & x_1 & +x_2 & +x_3 & +x_4 & & & & = 4 & \\
& x_1 & & & & +x_5 & & & = 2 & \\
& & & x_3 & & & +x_6 & & = 3 & \\
& & 3x_2 & +x_3 & & & & +x_7 & = 6 & \\
& & & & & & & x_j & \geq 0 & ,\forall j
\end{array}
$$

$$
(A|b) = \begin{pmatrix} 1 & 1 & 1 & 1 & & & & | & 4 \\ 1 & & & & 1 & & & | & 2 \\ & & 1 & & & 1 & & | & 3 \\ & 3 & 1 & & & & 1 & | & 6 \end{pmatrix}
$$

A has full row rank $m = 4$.

Example:

Basis 1:

$$\left(\begin{array}{ccccccc|c} 1 & 1 & 1 & 1 & & & & 4 \\ 1 & & & & 1 & & & 2 \\ & & 1 & & & 1 & & 3 \\ & 3 & 1 & & & & 1 & 6 \end{array}\right)$$

Basis 2:

$$\left(\begin{array}{ccccccc|c} 1 & 1 & 1 & 1 & & & & 4 \\ 1 & & & & 1 & & & 2 \\ & & 1 & & & 1 & & 3 \\ & 3 & 1 & & & & 1 & 6 \end{array}\right)$$

Basis 3:

$$\left(\begin{array}{ccccccc|c} 1 & 1 & 1 & 1 & & & & 4 \\ 1 & & & & 1 & & & 2 \\ & & 1 & & & 1 & & 3 \\ & 3 & 1 & & & & 1 & 6 \end{array}\right)$$

Example:

- Every basis B is invertible and can be transformed into the identity matrix by elementary row operations and column permutations. (Gaussian elemination)
- If we transform the whole expended matrix with these operations, we obtain a solution of $Ax = b$ by setting the basic variables to the transformed right-hand-side. Such a solution is called basic solution for basis B.

Basis 1:

$$\left(\begin{array}{ccccccc|c} 1 & 1 & 1 & 1 & & & & 4 \\ 1 & & & & 1 & & & 2 \\ & & 1 & & & 1 & & 3 \\ & 3 & 1 & & & & 1 & 6 \end{array}\right)$$

Example:

Basis 2:

$$\left(\begin{array}{ccccccc|c}
1 & 1 & 1 & 1 & & & & 4 \\
1 & & & & 1 & & & 2 \\
 & & 1 & & & 1 & & 3 \\
 & 3 & 1 & & & & 1 & 6
\end{array}\right)$$

Example:

Basis 3:

$$\left(\begin{array}{ccccccc|c} 1 & 1 & 1 & 1 & & & & 4 \\ 1 & & & & 1 & & & 2 \\ & & 1 & & & 1 & & 3 \\ & 3 & 1 & & & & 1 & 6 \end{array}\right)$$

Example:

- If we permute the columns of A and x such that $A = (A_B, A_N)$ and $x = \binom{x_B}{x_N}$, then the elementary transformations correspond to multiplying the linear system

$$(A_B, A_N)\begin{pmatrix} x_B \\ x_N \end{pmatrix} = b$$

from the left with the inverse B^{-1} of the basis:

$$\begin{aligned} & & B^{-1}(A_B, A_N)\begin{pmatrix} x_B \\ x_N \end{pmatrix} &= B^{-1}b \\ &\Leftrightarrow & B^{-1}A_B x_B + B^{-1}A_N x_N &= B^{-1}b \\ &\Leftrightarrow & x_B + B^{-1}A_N x_N &= B^{-1}b \end{aligned}$$

- Setting $x_N = 0$, we obtain $x_B = B^{-1}b$.
- So if B is a basis, we obtain the associated basic solution $x = (x_B, x_N)^T$ as $x_B = B^{-1}b$, $x_N = 0$.

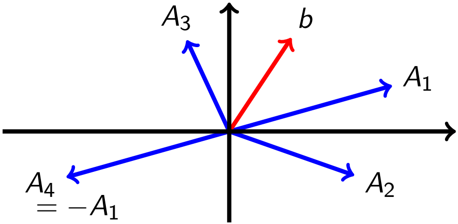
Basic Columns and Basic Solutions

Observation 3.15.

Let $x \in \mathbb{R}^n$ be a basic solution, then:

- $B \cdot x_B = b$ and thus $x_B = B^{-1} \cdot b$;
- x is a basic feasible solution if and only if $x_B = B^{-1} \cdot b \geq 0$.

Example: $m = 2$



- A_1, A_3 or A_2, A_3 form bases with corresp. basic feasible solutions.
- A_1, A_4 do not form a basis.
- A_1, A_2 and A_2, A_4 and A_3, A_4 form bases with infeasible basic solution.

Bases and Basic Solutions

Corollary 3.16.

- Every basis $A_{B(1)}, \dots, A_{B(m)}$ determines a unique basic solution.
- Thus, different basic solutions correspond to different bases.
- But: two different bases might yield the same basic solution.

Example: If $b = 0$, then $x = 0$ is the only basic solution.

Adjacent Bases

Definition 3.17.

Two bases $A_{B(1)}, \ldots, A_{B(m)}$ and $A_{B'(1)}, \ldots, A_{B'(m)}$ are adjacent if they share all but one column.

Observation 3.18.

a. Two adjacent basic solutions can always be obtained from two adjacent bases.
b. If two adjacent bases lead to distinct basic solutions, then the latter are adjacent.

Degeneracy

Definition 3.19.

A basic solution x of a polyhedron P is degenerate if more than n constraints are active at x.

Observation 3.20.

Let $P = \{x \in \mathbb{R}^n \mid A \cdot x = b,\ x \geq 0\}$ be a polyhedron in standard form with $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

- a) A basic solution $x \in P$ is degenerate if and only if more than $n - m$ components of x are zero.
- b) For a non-degenerate basic solution $x \in P$, there is a unique basis.

Three Different Reasons for Degeneracy

i. redundant variables

Example:

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_3 &= 0 \\ x_1,\ x_2,\ x_3 &\geq 0 \end{aligned} \quad \longleftrightarrow \quad A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

ii. redundant constraints

Example:

$$\begin{aligned} x_1 + 2\,x_2 &\leq 3 \\ 2\,x_1 + x_2 &\leq 3 \\ x_1 + x_2 &\leq 2 \\ x_1,\ x_2 &\geq 0 \end{aligned}$$

iii. geometric reasons

Example: Octahedron

Observation 3.21.

Perturbing the right hand side vector b may remove degeneracy.

Existence of Extreme Points

Definition 3.22.

A polyhedron $P \subseteq \mathbb{R}^n$ contains a line if there is $x \in P$ and a direction $d \in \mathbb{R}^n \setminus \{0\}$ such that

$$x + \lambda \cdot d \in P \qquad \text{for all } \lambda \in \mathbb{R}.$$

Theorem 3.23.

Let $P = \{x \in \mathbb{R}^n \mid A \cdot x \geq b\} \neq \emptyset$ with $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. The following are equivalent:

- i There exists an extreme point $x \in P$.
- ii P does not contain a line.
- iii A contains n linearly independent rows.

Existence of Extreme Points (cont.)

Corollary 3.24.

a. A non-empty polytope contains an extreme point.

b. A non-empty polyhedron in standard form contains an extreme point.

Proof of b:

$$\begin{aligned} A \cdot x &= b \\ x &\geq 0 \end{aligned} \qquad \longleftrightarrow \qquad \begin{pmatrix} A \\ -A \\ I \end{pmatrix} \cdot x \geq \begin{pmatrix} b \\ -b \\ 0 \end{pmatrix}$$

□

Example:

$$P = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{array}{rcrl} x_1 & + & x_2 & \geq 1 \\ x_1 & + & 2\,x_2 & \geq 0 \end{array} \right\}$$

contains a line since $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in P$ for all $\lambda \in \mathbb{R}$.

Optimality of Extreme Points

Theorem 3.25.

Let $P \subseteq \mathbb{R}^n$ a polyhedron and $c \in \mathbb{R}^n$. If P has an extreme point and $\min\{c^T \cdot x \mid x \in P\}$ is bounded, there is an extreme point that is optimal.

Corollary 3.26.

Every linear programming problem is either infeasible or unbounded or there exists an optimal solution.

Proof: Every linear program is equivalent to an LP in standard form. The claim thus follows from Corollary 3.24 and Theorem 3.25. □