

COMP331/557

Chapter 4:  
The Simplex Method

(Bertsimas & Tsitsiklis, Chapter 3)

## Linear Program in Standard Form

Throughout this chapter, we consider the following standard form problem:

$$\begin{aligned} & \text{minimize} && c^T \cdot x \\ & \text{subject to} && A \cdot x = b \\ & && x \geq 0 \end{aligned}$$

with  $A \in \mathbb{R}^{m \times n}$ ,  $\text{rank}(A) = m$ ,  $b \in \mathbb{R}^m$ , and  $c \in \mathbb{R}^n$ .

Recall:

- ▶ Let  $B = (A_{B(1)}, \dots, A_{B(m)})$  be a basis matrix of  $A$ . Then  $B$  corresponds to the **basic solution**  $x = (x_B, x_N)^T$ , where  $x_B = B^{-1}b$  and  $x_N = 0$ .
- ▶  $x = (x_B, x_N)^T$  is a **basic feasible solution** if  $x_B \geq 0$ .

# Main Idea of the Simplex Method

## Idea

Change basis by exchanging one basic column with one non-basic column.

More precisely:

- ▶ Start with a basis  $B$  defining a system with **basic feasible solution**.
- ▶ Then proceed in iterations. In each iteration:
  - ▶ **select a nonbasic column  $j$**  such that bringing  $j$  into the basis **decreases** (or at least does not increase) the value of the **objective function**. Stop, if no such column exists.
  - ▶ select a basic column  $\ell$  such that exchanging columns  $j$  and  $\ell$  maintain a basis with associated **basic feasible solution**
  - ▶ update the corresponding system

Iterations are called **pivot steps**.



## Set of Feasible Solutions

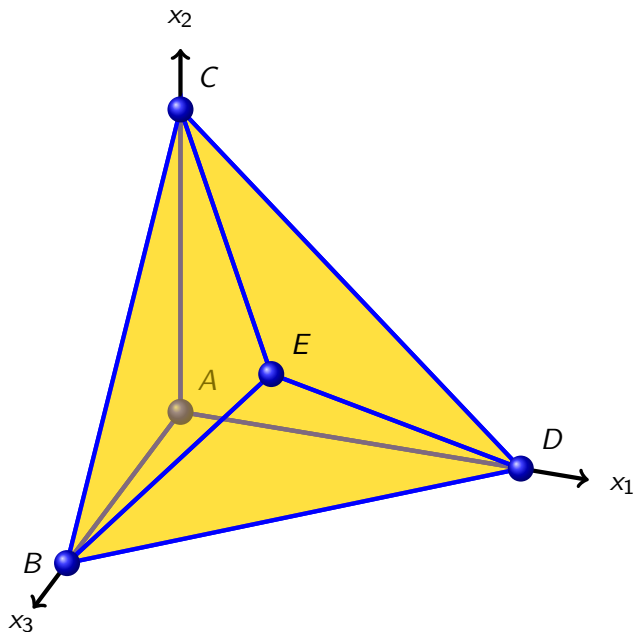
$$A = (0, 0, 0)^T$$

$$B = (0, 0, 10)^T$$

$$C = (0, 10, 0)^T$$

$$D = (10, 0, 0)^T$$

$$E = (4, 4, 4)^T$$



## Introducing Slack Variables

$$\begin{array}{llllll} \min & -10x_1 & - & 12x_2 & - & 12x_3 \\ \text{s.t.} & x_1 & + & 2x_2 & + & 2x_3 & \leq & 20 \\ & 2x_1 & + & x_2 & + & 2x_3 & \leq & 20 \\ & 2x_1 & + & 2x_2 & + & x_3 & \leq & 20 \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

### LP in standard form

$$\begin{array}{llllllllll} \min & -10x_1 & - & 12x_2 & - & 12x_3 & & & & & \\ \text{s.t.} & x_1 & + & 2x_2 & + & 2x_3 & + & x_4 & & & = & 20 \\ & 2x_1 & + & x_2 & + & 2x_3 & & & + & x_5 & = & 20 \\ & 2x_1 & + & 2x_2 & + & x_3 & & & & + & x_6 & = & 20 \\ & & & & & & & & & & x_1, \dots, x_6 & \geq & 0 \end{array}$$

### Observation

The right hand side of the system is non-negative. Therefore the point  $(0, 0, 0, 20, 20, 20)^T$  is a basic feasible solution and we can start the simplex method with basis  $B(1) = 4, B(2) = 5, B(3) = 6$ .

## Setting Up the Simplex Tableau

$$\begin{array}{rcllclcl}
 \min & -10x_1 & -12x_2 & -12x_3 & & & \\
 \text{s.t.} & x_1 & +2x_2 & +2x_3 & +x_4 & & = 20 \\
 & 2x_1 & +x_2 & +2x_3 & & +x_5 & = 20 \\
 & 2x_1 & +2x_2 & +x_3 & & & +x_6 = 20 \\
 & & & & & & x_1, \dots, x_6 \geq 0
 \end{array}$$

with basic feasible solution:  $\underbrace{x_1 = x_2 = x_3 = 0}_{\text{non-basic variables}}, \underbrace{x_4 = 20, x_5 = 20, x_6 = 20}_{\text{basic variables}}$ .

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$
0	-10	-12	-12	0	0	0
$x_4 =$	20	1	2	2	1	0
$x_5 =$	20	2	1	2	0	1
$x_6 =$	20	2	2	1	0	0

**Remark:** Initialisation not always that easy. See next week.

## Pivoting

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	
	0	-10	-12	-12	0	0	0
$x_4 =$	20	1	2	2	1	0	0
$x_5 =$	20	2	1	2	0	1	0
$x_6 =$	20	2	2	1	0	0	1

- ▶ Determine **pivot column**
  - ▶ Which non-basic variable can we increase to improve objective value?



# Pivoting

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_1 \leq \frac{x_{B(i)}}{u_i}$
	0	-10	-12	0	0	0	
$x_4 =$	20	1	2	2	1	0	$\Rightarrow x_1 \leq 20$
$x_5 =$	20	2	1	2	0	1	$\Rightarrow x_1 \leq 10$
$x_6 =$	20	2	2	1	0	1	$\Rightarrow x_1 \leq 10$

- ▶ Determine **pivot column**
  - ▶ Which non-basic variable can we increase to improve objective value?
  - ▶ E. g., smallest subscript rule:  $\bar{c}_1 < 0$  and  $x_1$  enters the basis.
- ▶ Find **pivot row**. How large can we make  $x_1$  and stay feasible?
  - ▶ Rows 2 and 3 both attain the minimum.

# Pivoting

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_1 \leq \frac{x_{B(i)}}{u_i}$
	0	-10	-12	0	0	0	
$x_4 =$	20	1	2	1	0	0	$\Rightarrow x_1 \leq 20$
$x_5 =$	20	2	1	0	1	0	$\Rightarrow x_1 \leq 10$
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  - ▶ Choose  $i = 2$  with  $B(i) = 5$ .  $\implies x_5$  leaves the basis.

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  - ▶ Choose  $i = 2$  with  $B(i) = 5$ .  $\implies x_5$  leaves the basis.
- ▶ Perform basis change: Eliminate other entries in the **pivot column**.

## Pivoting

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$
100	0	-7	-2	0	5	0
$x_4 =$	20	1	2	2	1	0
$x_5 =$	20	2	1	2	0	1
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  - ▶ Rows 2 and 3 both attain the minimum.
  - ▶ Choose  $i = 2$  with  $B(i) = 5$ .  $\implies x_5$  leaves the basis.
- ▶ Perform basis change: Eliminate other entries in the **pivot column**.
- ▶ Obtain new basic feasible solution  $(10, 0, 0, 10, 0, 0)^T$  with cost -100.

## Geometric Interpretation in the Original Polyhedron

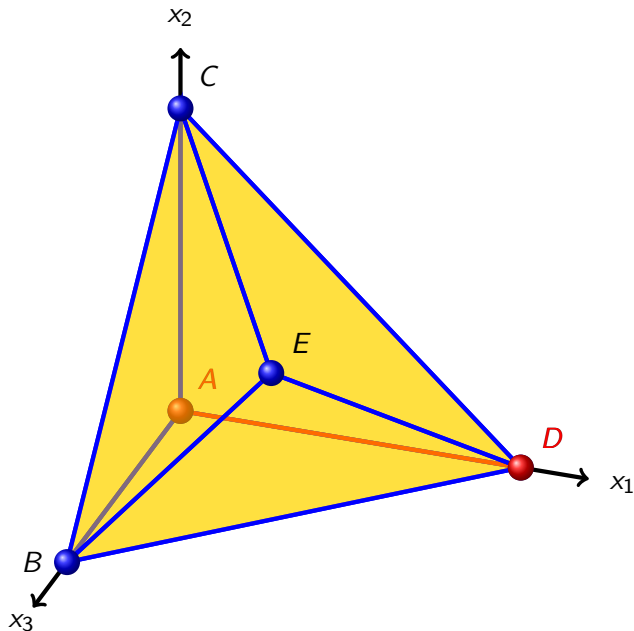
$$A = (0, 0, 0)^T$$

$$B = (0, 0, 10)^T$$

$$C = (0, 10, 0)^T$$

$$D = (10, 0, 0)^T$$

$$E = (4, 4, 4)^T$$



## New View of our Optimisation Problem

		$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$
	100	0	-7	-2	0	5	0
$x_4 =$	10	0	1.5	1	1	-0.5	0
$x_1 =$	10	1	0.5	1	0	0.5	0
$x_6 =$	0	0	1	-1	0	-1	1

### Corresponding Transformed LP

$$\begin{array}{ll}
 \min & -100 \quad -7x_2 \quad -2x_3 \quad \quad +5x_5 \\
 \text{s.t.} & \quad +1.5x_2 \quad +x_3 \quad +x_4 \quad -0.5x_5 \quad = 10 \\
 & \quad x_1 \quad +0.5x_2 \quad +x_3 \quad \quad +0.5x_5 \quad = 10 \\
 & \quad \quad +x_2 \quad -x_3 \quad \quad -x_5 \quad +x_6 \quad = 0 \\
 & \quad \quad \quad \quad \quad \quad \quad \quad x_1, \dots, x_6 \geq 0
 \end{array}$$

with basic feasible solution:  $\underbrace{x_2 = x_3 = x_5 = 0}_{\text{non-basic variables}}, \underbrace{x_4 = 10, x_1 = 10, x_6 = 0}_{\text{basic variables}}$ .

# Equivalence

Property: Transformed problem is exactly the same as original one.

## Original LP:

$$\begin{array}{llllllll} \min & -10 x_1 & -12 x_2 & -12 x_3 & & & & \\ \text{s.t.} & x_1 & +2 x_2 & +2 x_3 & +x_4 & & & = 20 \\ & 2 x_1 & +x_2 & +2 x_3 & & +x_5 & & = 20 \\ & 2 x_1 & +2 x_2 & +x_3 & & & +x_6 & = 20 \\ & & & & & & & x_1, \dots, x_6 \geq 0 \end{array}$$

Update rule:  $x_1 \mapsto 10 - 0.5 x_2 - x_3 - 0.5 x_5$

## Transformed LP

$$\begin{array}{llllllll} \min & -100 & & -7 x_2 & -2 x_3 & & +5 x_5 & \\ \text{s.t.} & & & +1.5 x_2 & +x_3 & +x_4 & -0.5 x_5 & = 10 \\ & x_1 & & +0.5 x_2 & +x_3 & & +0.5 x_5 & = 10 \\ & & & +x_2 & -x_3 & & -x_5 & +x_6 = 0 \\ & & & & & & & x_1, \dots, x_6 \geq 0 \end{array}$$

## Next Iterations

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$
	100	0	-7	0	5	0
$x_4 =$	10	0	1.5	1	-0.5	0
$x_1 =$	10	1	0.5	0	0.5	0
$x_6 =$	0	0	1	0	-1	1

►  $\bar{c}_2, \bar{c}_3 < 0 \implies$  two possible choices for pivot column.

## Next Iterations

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$\frac{x_{B(i)}}{u_i}$
	100	0	-7	-2	0	5	0
$x_4 =$	10	0	1.5	1	-0.5	0	10
$x_1 =$	10	1	0.5	1	0.5	0	10
$x_6 =$	0	0	1	-1	-1	1	-

- ▶  $\bar{c}_2, \bar{c}_3 < 0 \implies$  two possible choices for **pivot column**.
- ▶ Choose  $x_3$  to enter the new basis.
- ▶  $u_3 < 0 \implies$  third row cannot be chosen as **pivot row**.
- ▶ Choose  $x_4$  to leave basis.



## Next Iterations

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$
100	0	-7	-2	0	5	0
$x_4 =$	10	0	1.5	1	-0.5	0
$x_1 =$	10	1	0.5	1	0.5	0
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- ▶ Choose  $x_3$  to enter the new basis.
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- ▶ Choose  $x_4$  to leave basis.

## Next Iterations

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	
	120	0	-4	0	2	4	0
$x_4 =$	10	0	1.5	1	1	-0.5	0
$x_1 =$	10	1	0.5	1	0	0.5	0
$x_6 =$	0	0	1	-1	0	-1	1

- ▶  $\bar{c}_2, \bar{c}_3 < 0 \implies$  two possible choices for **pivot column**.
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	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	
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## Next Iterations

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	
	120	0	-4	0	2	4	0
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## Next Iterations

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	
	120	0	-4	0	2	4	0
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## Next Iterations

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- ▶ Choose  $x_3$  to enter the new basis.
- ▶  $u_3 < 0 \implies$  third row cannot be chosen as **pivot row**.
- ▶ Choose  $x_4$  to leave basis.
- ▶ New basic feasible solution  $(0, 0, 10, 0, 0, 10)^T$  with **cost -120**, corresponding to point  $B$  in the original polyhedron.

## Geometric Interpretation in the Original Polyhedron

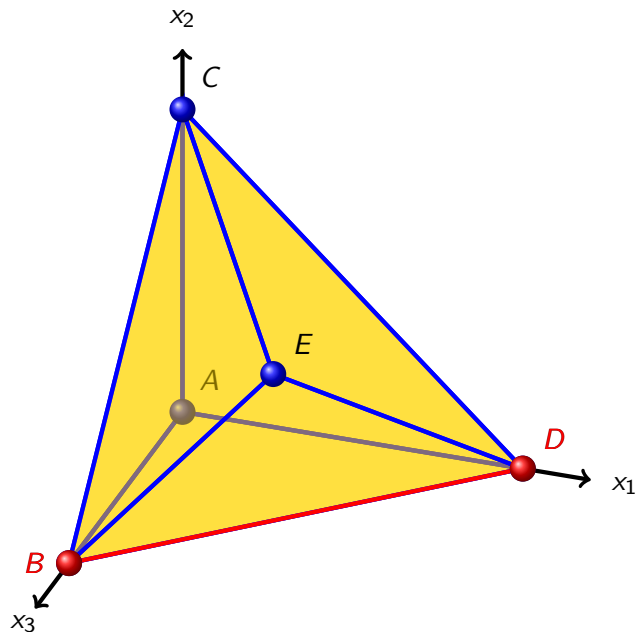
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$$E = (4, 4, 4)^T$$





## Next Iterations

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$\frac{x_{B(i)}}{u_i}$
	120	0	-4	0	2	4	0
$x_3 =$	10	0	1.5	1	1	-0.5	0
$x_1 =$	0	1	-1	0	-1	1	0
$x_6 =$	10	0	2.5	0	1	-1.5	1

## Next Iterations

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$\frac{x_{B(i)}}{u_i}$
	120	0	-4	0	2	4	0
$x_3 =$	10	0	1.5	1	1	-0.5	0
$x_1 =$	0	1	-1	0	-1	1	0
$x_6 =$	10	0	2.5	0	1	-1.5	1

4 <  $\frac{20}{3}$

## Next Iterations

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	
120	0	-4	0	2	4	0	
$x_3 =$	10	0	1.5	1	-0.5	0	
$x_1 =$	0	1	-1	0	-1	1	
$x_6 =$	10	0	2.5	0	1	-1.5	1

$x_2$  enters the basis,  $x_6$  leaves it. We get

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	
136	0	0	0	3.6	1.6	1.6	
$x_3 =$	4	0	0	1	0.4	0.4	-0.6
$x_1 =$	4	1	0	0	-0.6	0.4	0.4
$x_2 =$	4	0	1	0	0.4	-0.6	0.4

and the reduced costs are all non-negative.

Thus  $(4, 4, 4, 0, 0, 0)$  is an optimal solution with cost -136, corresponding to point  $E = (4, 4, 4)$  in the original polyhedron. **Why is this optimal?**

## All Iterations from Geometric Point of View

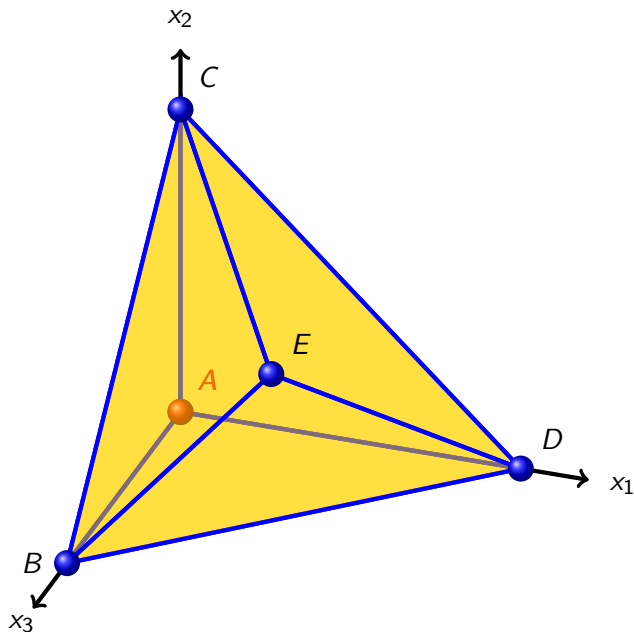
$$A = (0, 0, 0)^T$$

$$B = (0, 0, 10)^T$$

$$C = (0, 10, 0)^T$$

$$D = (10, 0, 0)^T$$

$$E = (4, 4, 4)^T$$



## All Iterations from Geometric Point of View

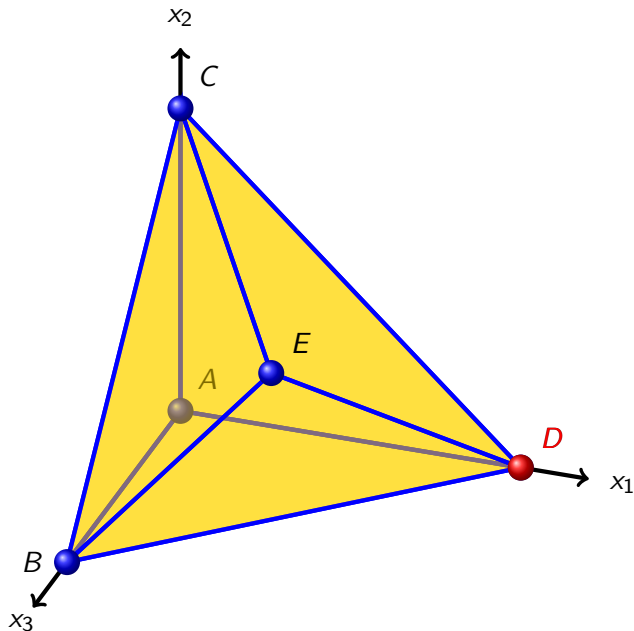
$$A = (0, 0, 0)^T$$

$$B = (0, 0, 10)^T$$

$$C = (0, 10, 0)^T$$

$$D = (10, 0, 0)^T$$

$$E = (4, 4, 4)^T$$



## All Iterations from Geometric Point of View

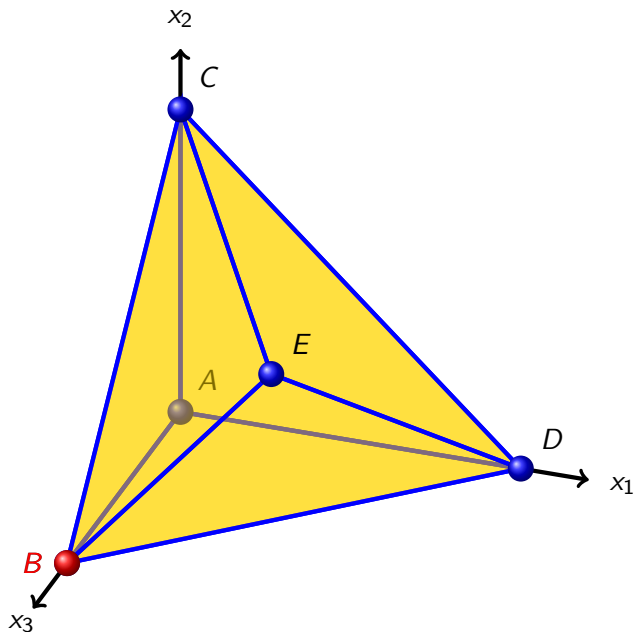
$$A = (0, 0, 0)^T$$

$$B = (0, 0, 10)^T$$

$$C = (0, 10, 0)^T$$

$$D = (10, 0, 0)^T$$

$$E = (4, 4, 4)^T$$



## All Iterations from Geometric Point of View

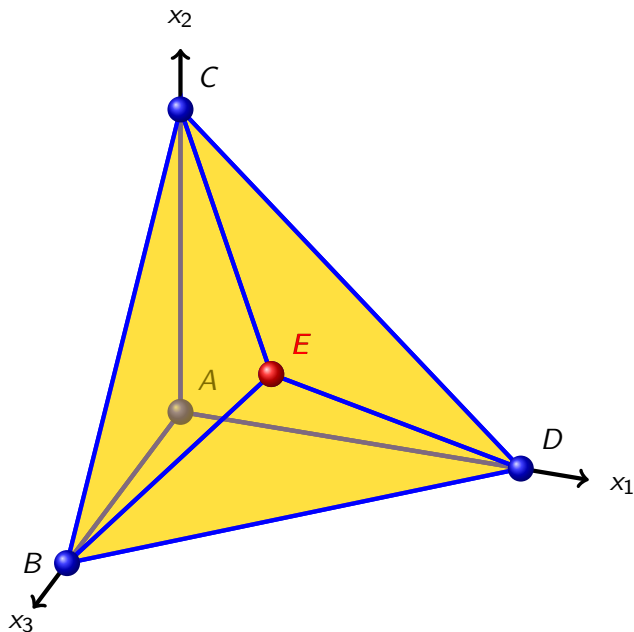
$$A = (0, 0, 0)^T$$

$$B = (0, 0, 10)^T$$

$$C = (0, 10, 0)^T$$

$$D = (10, 0, 0)^T$$

$$E = (4, 4, 4)^T$$



## Cycling

**Problem:** If an LP is degenerate, the simplex method might end up in an infinite loop (**cycling**).

**Example:**

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$
	3	$-3/4$	20	$-1/2$	6	0	0
$x_5 =$	0	$1/4$	-8	-1	9	1	0
$x_6 =$	0	$1/2$	-12	$-1/2$	3	0	1
$x_7 =$	1	0	0	1	0	0	1

## Pivoting rules

- ▶ **Column selection:** let nonbasic variable with most negative reduced cost  $\bar{c}_j$  enter the basis, i. e., **steepest descent rule**.
- ▶ **Row selection:** among basic variables that are eligible to exit the basis, select the one with **smallest subscript**.



## Iteration 1

:

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$\frac{x_{B(i)}}{u_i}$
	3	$-3/4$	20	$-1/2$	6	0	0	0
$x_5 =$	0	1/4	-8	-1	9	1	0	0
$x_6 =$	0	1/2	-12	$-1/2$	3	0	1	0
$x_7 =$	1	0	0	1	0	0	1	-

Basis change:  $x_1$  enters the basis  $x_5$  leaves.

Bases visited

(5, 6, 7)

## Iteration 1

:

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$
3	$-3/4$	20	$-1/2$	6	0	0	0
$x_5 =$	0	$1/4$	-8	-1	9	1	0
$x_6 =$	0	$1/2$	-12	$-1/2$	3	0	1
$x_7 =$	1	0	0	1	0	0	1

Basis change:  $x_1$  enters the basis  $x_5$  leaves.

Bases visited

(5, 6, 7)

## Iteration 1

:

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$
3	0	-4	$-7/2$	33	3	0	0
$x_5 =$	0	1/4	-8	-1	9	1	0
$x_6 =$	0	1/2	-12	$-1/2$	3	0	1
$x_7 =$	1	0	0	1	0	0	1

Basis change:  $x_1$  enters the basis  $x_5$  leaves.

Bases visited

(5, 6, 7)

## Iteration 1

:

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	
	3	0	-4	$-7/2$	33	3	0	0
$x_5 =$	0	1/4	-8	-1	9	1	0	0
$x_6 =$	0	1/2	-12	$-1/2$	3	0	1	0
$x_7 =$	1	0	0	1	0	0	0	1

Basis change:  $x_1$  enters the basis  $x_5$  leaves.

Bases visited

(5, 6, 7)

## Iteration 1

:

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$
3	0	-4	$-7/2$	33	3	0	0
$x_5 =$	0	1/4	-8	-1	9	1	0
$x_6 =$	0	0	4	$3/2$	-15	-2	0
$x_7 =$	1	0	0	1	0	0	1

Basis change:  $x_1$  enters the basis  $x_5$  leaves.

Bases visited

(5, 6, 7)

## Iteration 1

:

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$
3	0	-4	$-7/2$	33	3	0	0
$x_5 =$	0	1/4	-8	-1	9	1	0
$x_6 =$	0	0	4	$3/2$	-15	-2	1
$x_7 =$	1	0	0	1	0	0	1

Basis change:  $x_1$  enters the basis  $x_5$  leaves.

Bases visited

(5, 6, 7)

## Iteration 1

:

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$
3	0	-4	$-7/2$	33	3	0	0
$x_5 =$	0	1/4	-8	-1	9	1	0
$x_6 =$	0	0	4	$3/2$	-15	-2	1
$x_7 =$	1	0	0	1	0	0	1

Basis change:  $x_1$  enters the basis  $x_5$  leaves.

Bases visited

(5, 6, 7)

## Iteration 1

:

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$
$x_1 =$	0	-4	$-7/2$	33	3	0	0
$x_6 =$	0	4	$3/2$	-15	-2	1	0
$x_7 =$	1	0	1	0	0	0	1

Basis change:  $x_1$  enters the basis  $x_5$  leaves.

Bases visited

(5, 6, 7)



## Iteration 2

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$\frac{x_{B(i)}}{u_i}$
3	0	-4	$-7/2$	33	3	0	0	
$x_1 =$	0	1	-32	-4	36	4	0	-
$x_6 =$	0	0	4	$3/2$	-15	-2	1	0
$x_7 =$	1	0	0	1	0	0	0	-

Basis change:  $x_2$  enters the basis  $x_6$  leaves.

Bases visited

$(5, 6, 7) \rightarrow (1, 6, 7)$

## Iteration 3

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$\frac{x_{B(i)}}{u_i}$
3	0	0	-2	18	1	1	0	
$x_1 =$	0	1	0	8	-84	-12	8	0
$x_2 =$	0	0	1	3/8	-15/4	-1/2	1/4	0
$x_7 =$	1	0	0	1	0	0	0	1

Basis change:  $x_3$  enters the basis  $x_1$  leaves.

### Bases visited

$(5, 6, 7) \rightarrow (1, 6, 7) \rightarrow (1, 2, 7)$

## Iteration 4

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$\frac{x_{B(i)}}{u_i}$	
3	$1/4$	0	0	-3	-2	3	0		
$x_3 =$	0	$1/8$	0	1	$-21/2$	$-3/2$	1	0	-
$x_2 =$	0	$-3/64$	1	0	<b><math>3/16</math></b>	$1/16$	$-1/8$	0	0
$x_7 =$	1	$-1/8$	0	0	$21/2$	$3/2$	-1	1	$2/21$

Basis change:  $x_4$  enters the basis  $x_2$  leaves.

### Bases visited

$(5, 6, 7) \rightarrow (1, 6, 7) \rightarrow (1, 2, 7) \rightarrow (3, 2, 7)$

## Iteration 5

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$\frac{x_{B(i)}}{u_i}$
	3	-1/2	16	0	0	-1	1	0
$x_3 =$	0	-5/2	56	1	0	2	-6	0
$x_4 =$	0	-1/4	16/3	0	1	1/3	-2/3	0
$x_7 =$	1	5/2	-56	0	0	-2	6	-

Basis change:  $x_5$  enters the basis  $x_3$  leaves.

### Bases visited

$(5, 6, 7) \rightarrow (1, 6, 7) \rightarrow (1, 2, 7) \rightarrow (3, 2, 7) \rightarrow (3, 4, 7)$

### Observation

After 4 pivoting iterations our basic feasible solution still has not changed.

## Iteration 6

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$\frac{x_{B(i)}}{u_i}$
	3	$-7/4$	44	$1/2$	0	0	-2	0
$x_5 =$	0	$-5/4$	28	$1/2$	0	1	-3	-
$x_4 =$	0	$1/6$	-4	$-1/6$	1	0	<b><math>1/3</math></b>	0
$x_7 =$	1	0	0	1	0	0	0	-

Basis change:  $x_6$  enters the basis  $x_4$  leaves.

### Bases visited

$(5, 6, 7) \rightarrow (1, 6, 7) \rightarrow (1, 2, 7) \rightarrow (3, 2, 7) \rightarrow (3, 4, 7)$   
 $\rightarrow (5, 4, 7)$

## Back at the Beginning

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$
3	$-3/4$	20	$-1/2$	6	0	0	0
$x_5 =$	0	$1/4$	-8	-1	9	1	0
$x_6 =$	0	$1/2$	-12	$-1/2$	3	0	1
$x_7 =$	1	0	0	1	0	0	1

### Bases visited

$(5, 6, 7) \rightarrow (1, 6, 7) \rightarrow (1, 2, 7) \rightarrow (3, 2, 7) \rightarrow (3, 4, 7)$   
 $\rightarrow (5, 4, 7) \rightarrow (5, 6, 7)$

This is the same basis that we started with.

### Conclusion

Continuing with the pivoting rules we agreed on at the beginning, the simplex method will never terminate in this example.

## Anticycling – Bland's Rule

We now discuss a pivoting rule that is guaranteed to avoid cycling:

### Smallest subscript pivoting rule (Bland's rule)

- 1 Choose the column  $A_j$  with  $\bar{c}_j < 0$  and  $j$  minimal to enter the basis.
- 2 Among all basic variables  $x_i$  that could exit the basis, select the one with smallest  $i$ .

### Theorem (without proof)

The simplex algorithm with Bland's rule does not cycle and thus terminates after a finite number of iterations.

## Finding an Initial Basic Feasible Solution

So far we always assumed that the simplex algorithm starts with a basic feasible solution. We now discuss how such a solution can be obtained.

- ▶ Introducing artificial variables
- ▶ The two-phase simplex method
- ▶ The big- $M$  method







## Initial Tableau

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$
	0	0	0	0	1	1	1	1
$x_5 =$	3	1	2	3	0	1	0	0
$x_6 =$	2	-1	2	6	0	0	1	0
$x_7 =$	5	0	4	9	0	0	0	1
$x_8 =$	1	0	0	3	1	0	0	1

Calculate reduced costs by eliminating the nonzero-entries for the basis-variables.

## Initial Tableau

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$
	0	0	0	0	1	1	1	1
$x_5 =$	3	1	2	3	0	1	0	0
$x_6 =$	2	-1	2	6	0	0	1	0
$x_7 =$	5	0	4	9	0	0	1	0
$x_8 =$	1	0	0	3	1	0	0	1

Calculate reduced costs by eliminating the nonzero-entries for the basis-variables.

## Initial Tableau

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	
	-3	-1	-2	-3	0	0	1	1	1
$x_5 =$	3	1	2	3	0	1	0	0	0
$x_6 =$	2	-1	2	6	0	0	1	0	0
$x_7 =$	5	0	4	9	0	0	0	1	0
$x_8 =$	1	0	0	3	1	0	0	0	1

Calculate reduced costs by eliminating the nonzero-entries for the basis-variables.

## Initial Tableau

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$
	-5	0	-4	-9	0	0	1	1
$x_5 =$	3	1	2	3	0	1	0	0
$x_6 =$	2	-1	2	6	0	0	1	0
$x_7 =$	5	0	4	9	0	0	1	0
$x_8 =$	1	0	0	3	1	0	0	1

Calculate reduced costs by eliminating the nonzero-entries for the basis-variables.

## Initial Tableau

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$
	-10	0	-8	-18	0	0	0	1
$x_5 =$	3	1	2	3	0	1	0	0
$x_6 =$	2	-1	2	6	0	0	1	0
$x_7 =$	5	0	4	9	0	0	0	1
$x_8 =$	1	0	0	3	1	0	0	1

Calculate reduced costs by eliminating the nonzero-entries for the basis-variables.

## Initial Tableau

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$
	-11	0	-8	-21	-1	0	0	0
$x_5 =$	3	1	2	3	0	1	0	0
$x_6 =$	2	-1	2	6	0	0	1	0
$x_7 =$	5	0	4	9	0	0	0	1
$x_8 =$	1	0	0	3	1	0	0	1

Calculate reduced costs by eliminating the nonzero-entries for the basis-variables.

Now we can proceed as seen before...



## Minimizing the Auxiliary Problem

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$
$-11$	0	-8	-21	-1	0	0	0	0
$x_5 =$	3	1	2	3	0	1	0	0
$x_6 =$	2	-1	2	6	0	0	1	0
$x_7 =$	5	0	4	9	0	0	0	1
$x_8 =$	1	0	0	3	1	0	0	1

Basis change:  $x_4$  enters the basis,  $x_8$  exits.

## Minimizing the Auxiliary Problem

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$
$-10$	0	-8	-18	0	0	0	0	1
$x_5 =$	3	1	2	3	0	1	0	0
$x_6 =$	2	-1	2	6	0	0	1	0
$x_7 =$	5	0	4	9	0	0	0	1
$x_4 =$	1	0	0	3	1	0	0	1

Basis change:  $x_3$  enters the basis,  $x_4$  exits.

## Minimizing the Auxiliary Problem

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$
$-4$	0	$-8$	0	6	0	0	0	7
$x_5 =$	2	1	2	0	$-1$	1	0	$-1$
$x_6 =$	0	$-1$	2	0	$-2$	0	1	$-2$
$x_7 =$	2	0	4	0	$-3$	0	0	$-3$
$x_3 =$	$1/3$	0	0	1	$1/3$	0	0	$1/3$

Basis change:  $x_2$  enters the basis,  $x_6$  exits.

## Minimizing the Auxiliary Problem

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$
$-4$	$-4$	$0$	$0$	$-2$	$0$	$4$	$0$	$-1$
$x_5 =$	$2$	$2$	$0$	$0$	$1$	$1$	$-1$	$0$
$x_2 =$	$0$	$-1/2$	$1$	$0$	$-1$	$0$	$1/2$	$0$
$x_7 =$	$2$	$2$	$0$	$0$	$1$	$0$	$-2$	$1$
$x_3 =$	$1/3$	$0$	$0$	$1$	$1/3$	$0$	$0$	$1/3$

Basis change:  $x_1$  enters the basis,  $x_5$  exits.

## Minimizing the Auxiliary Problem

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	
	0	0	0	0	2	2	0	1	
$x_1 =$	1	1	0	0	$1/2$	$1/2$	$-1/2$	0	$1/2$
$x_2 =$	$1/2$	0	1	0	$-3/4$	$1/4$	$1/4$	0	$-3/4$
$x_7 =$	0	0	0	0	0	-1	-1	1	0
$x_3 =$	$1/3$	0	0	1	$1/3$	0	0	0	$1/3$

Basic feasible solution for auxiliary problem with (auxiliary) cost value 0

⇒ Also feasible for the original problem - but not (yet) basic.

## Obtaining a Basis for the Original Problem

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	
$x_1 =$	0	0	0	0	2	2	0	1	
$x_2 =$	1	0	0	$1/2$	$1/2$	$-1/2$	0	$1/2$	
$x_7 =$	$1/2$	0	1	0	$-3/4$	$1/4$	$1/4$	0	$-3/4$
$x_3 =$	0	0	0	0	-1	-1	1	0	
	$1/3$	0	0	1	$1/3$	0	0	0	$1/3$

### Observation

Restricting the tableau to the original variables, we get a zero-row.

Thus the original equations are linearly dependent.

→ We can remove the third row.

## Obtaining a Basis for the Original Problem

	$x_1$	$x_2$	$x_3$	$x_4$	
0	1	1	1	0	
$x_1 =$	1	1	0	0	$1/2$
$x_2 =$	$1/2$	0	1	0	$-3/4$
$x_3 =$	$1/3$	0	0	1	$1/3$

We finally obtain a basic feasible solution for the original problem.

Computing the reduced costs for this basis:

- ▶ Put original objective function in row 0.
- ▶ Compute reduced costs by eliminating the nonzero entries for the basic variables.

The simplex method (phase II) can now start with its typical iterations.

## Obtaining a Basis for the Original Problem

	$x_1$	$x_2$	$x_3$	$x_4$
	$-11/6$	0	0	$-1/12$
$x_1 =$	1	1	0	$1/2$
$x_2 =$	$1/2$	0	1	$-3/4$
$x_3 =$	$1/3$	0	0	$1/3$

We finally obtain a basic feasible solution for the original problem.

Computing the reduced costs for this basis:

- ▶ Put original objective function in row 0.
- ▶ Compute reduced costs by eliminating the nonzero entries for the basic variables.

The simplex method (phase II) can now start with its typical iterations.





## Phase I of the Simplex Method

**Given:** LP in standard form:  $\min\{c^T \cdot x \mid A \cdot x = b, x \geq 0\}$

- 1 Transform problem such that  $b \geq 0$  (multiply constraints by  $-1$ ).
- 2 Introduce artificial variables  $y_1, \dots, y_m$  and solve auxiliary problem

$$\min \sum_{i=1}^m y_i \quad \text{s.t.} \quad A \cdot x + I_m \cdot y = b, \quad x, y \geq 0 .$$

- 3 If optimal cost is positive, then STOP (original LP is infeasible).
- 4 If no artificial variable is in final basis, eliminate artificial variables and columns and STOP (feasible basis for original LP has been found).
- 5 If  $\ell$ th basic variable is artificial, find  $j \in \{1, \dots, n\}$  with  $\ell$ th entry in  $B^{-1} \cdot A_j$  nonzero. Use this entry as pivot element and replace  $\ell$ th basic variable with  $x_j$ .
- 6 If no such  $j \in \{1, \dots, n\}$  exists, eliminate  $\ell$ th row (constraint).

# The Two-phase Simplex Method

## Two-phase simplex method

- 1 Given an LP in standard form, first run phase I.
- 2 If phase I yields a basic feasible solution for the original LP, enter “phase II” (see above).

## Possible outcomes of the two-phase simplex method

- i Problem is infeasible (detected in phase I).
- ii Problem is feasible but rows of  $A$  are linearly dependent (detected and corrected at the end of phase I by eliminating redundant constraints.)
- iii Optimal cost is  $-\infty$  (detected in phase II).
- iv Problem has optimal basic feasible solution (found in phase II).

**Remark:** (ii) is not an outcome but only an intermediate result leading to outcome (iii) or (iv).

## Big- $M$ Method

**Alternative idea:** Combine the two phases into one by introducing sufficiently large **penalty costs** for artificial variables.

This way, the LP

$$\begin{array}{ll} \min & \sum_{i=1}^n c_i x_i \\ \text{s.t.} & A \cdot x = b \\ & x \geq 0 \end{array}$$

becomes:

$$\begin{array}{ll} \min & \sum_{i=1}^n c_i x_i + M \cdot \sum_{j=1}^m y_j \\ \text{s.t.} & A \cdot x + I_m \cdot y = b \\ & x, y \geq 0 \end{array}$$

**Remark:** If  $M$  is sufficiently large and the original program has a feasible solution, all artificial variables will be driven to zero by the simplex method.

## How to Choose $M$ ?

### Observation

Initially,  $M$  only occurs in the zeroth row. As the zeroth row never becomes pivot row, this property is maintained while the simplex method is running.

All we need to have is an order on all values that can appear as reduced cost coefficients.

### Order on cost coefficients

$$aM + b < cM + d \quad :\iff \quad (a < c) \vee (a = c \wedge b < d)$$

In particular,  $-aM + b < 0 < aM + b$  for any positive  $a$  and arbitrary  $b$ , and we can decide whether a cost coefficient is negative or not.

→ There is no need to give  $M$  a fixed numerical value.

## Example

Example:

$$\begin{array}{rcccccccl} \min & x_1 & + & x_2 & + & x_3 & & & & \\ \text{s.t.} & x_1 & + & 2x_2 & + & 3x_3 & & & = & 3 \\ & -x_1 & + & 2x_2 & + & 6x_3 & & & = & 2 \\ & & & 4x_2 & + & 9x_3 & & & = & 5 \\ & & & & & 3x_3 & + & x_4 & = & 1 \\ & & & & & & & x_1, \dots, x_4 & \geq & 0 \end{array}$$



## Forming the Initial Tableau

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$
0	1	1	1	0	$M$	$M$	$M$
3	1	2	3	0	1	0	0
2	-1	2	6	0	0	1	0
5	0	4	9	0	0	0	1
1	0	0	3	1	0	0	0

Compute reduced costs by eliminating the nonzero entries for the basic variables.



## Forming the Initial Tableau

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$
0	1	1	1	0	$M$	$M$	$M$
3	1	2	3	0	1	0	0
2	-1	2	6	0	0	1	0
5	0	4	9	0	0	0	1
1	0	0	3	1	0	0	0

Compute reduced costs by eliminating the nonzero entries for the basic variables.

## Forming the Initial Tableau

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$
$-3M$	$-M + 1$	$-2M + 1$	$-3M + 1$	0	0	$M$	$M$
3	1	2	3	0	1	0	0
2	-1	2	6	0	0	1	0
5	0	4	9	0	0	0	1
1	0	0	3	1	0	0	0

Compute reduced costs by eliminating the nonzero entries for the basic variables.

## Forming the Initial Tableau

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$
$-5M$	1	$-4M + 1$	$-9M + 1$	0	0	0	$M$
3	1	2	3	0	1	0	0
2	-1	2	6	0	0	1	0
5	0	4	9	0	0	0	1
1	0	0	3	1	0	0	0

Compute reduced costs by eliminating the nonzero entries for the basic variables.

## Forming the Initial Tableau

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$
$-10M$	1	$-8M + 1$	$-18M + 1$	0	0	0	0
3	1	2	3	0	1	0	0
2	-1	2	6	0	0	1	0
5	0	4	9	0	0	0	1
1	0	0	3	1	0	0	0

Compute reduced costs by eliminating the nonzero entries for the basic variables.

## Forming the Initial Tableau

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$
$-10M$	1	$-8M + 1$	$-18M + 1$	0	0	0	0
3	1	2	3	0	1	0	0
2	-1	2	6	0	0	1	0
5	0	4	9	0	0	0	1
1	0	0	3	1	0	0	0

Compute reduced costs by eliminating the nonzero entries for the basic variables.

## First Iteration

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$
$-10M$	1	$-8M + 1$	$-18M + 1$	0	0	0	0
3	1	2	3	0	1	0	0
2	-1	2	6	0	0	1	0
5	0	4	9	0	0	0	1
1	0	0	3	1	0	0	0

Reduced costs for  $x_2$  and  $x_3$  are negative.

Basis change:  $x_3$  enters the basis,  $x_4$  leaves.

## Second Iteration

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$
$-4M - 1/3$	1	$-8M + 1$	0	$6M - 1/3$	0	0	0
2	1	2	0	-1	1	0	0
0	-1	2	0	-2	0	1	0
2	0	4	0	-3	0	0	1
$1/3$	0	0	1	$1/3$	0	0	0

Basis change:  $x_2$  enters the basis,  $x_6$  leaves.

## Third Iteration

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$
$-4M - 1/3$	$-4M + 3/2$	0	0	$-2M + 2/3$	0	$4M - 1/2$	0
2	2	0	0	1	1	-1	0
0	$-1/2$	1	0	-1	0	$1/2$	0
2	2	0	0	1	0	-2	1
$1/3$	0	0	1	$1/3$	0	0	0

Basis change:  $x_1$  enters the basis,  $x_5$  leaves.



## Fourth Iteration

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$
$-11/6$	0	0	0	$-1/12$	$2M - 3/4$	$2M + 1/4$	0
21	1	0	0	$1/2$	$1/2$	$-1/2$	0
$1/2$	0	1	0	$-3/4$	$1/4$	$1/4$	0
0	0	0	0	0	-1	-1	1
$1/3$	0	0	1	$1/3$	0	0	0

Note that all artificial variables have already been driven to 0.

Basis change:  $x_4$  enters the basis,  $x_3$  leaves.

## Fifth Iteration

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$
$-7/4$	0	0	$1/4$	0	$2M - 3/4$	$2M + 1/4$	0
$1/2$	1	0	$-3/2$	0	$1/2$	$-1/2$	0
$5/4$	0	1	$9/4$	0	$1/4$	$1/4$	0
0	0	0	0	0	-1	-1	1
1	0	0	3	1	0	0	0

We now have an optimal solution of the auxiliary problem, as all costs are nonnegative ( $M$  presumed large enough).

By eliminating the third row as in the previous example, we get a basic feasible and also optimal solution to the original problem.

# Computational Efficiency of the Simplex Method

## Observation

The computational efficiency of the simplex method is determined by

- i the computational effort of each iteration;
- ii the number of iterations.

**Question:** How many iterations are needed in the worst case?

## Idea for negative answer (lower bound)

Describe

- ▶ a polyhedron with an exponential number of vertices;
- ▶ a path that visits all vertices and always moves from a vertex to an adjacent one that has lower costs.



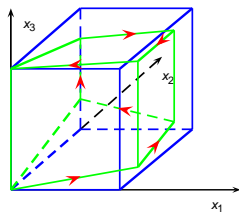
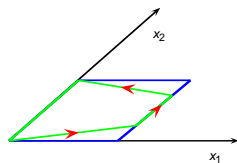
## Computational Efficiency of the Simplex Method (cont.)

### Klee-Minty cube

Consider a perturbation of the unit cube in  $\mathbb{R}^n$ , defined by the constraints

$$\begin{aligned} 0 &\leq x_1 \leq 1, \\ \epsilon x_{i-1} &\leq x_i \leq 1 - \epsilon x_{i-1}, \quad i = 2, \dots, n \end{aligned}$$

for some  $\epsilon \in (0, 1/2)$ .



## Computational Efficiency of the Simplex Method (cont.)

### Klee-Minty cube

$$0 \leq x_1 \leq 1,$$
$$\epsilon x_{i-1} \leq x_i \leq 1 - \epsilon x_{i-1}, \quad i = 2, \dots, n, \epsilon \in (0, 1/2)$$

### Theorem 4.1.

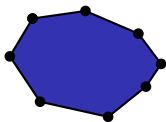
Consider the linear programming problem of minimizing  $-x_n$  subject to the constraints above. Then,

- a** the feasible set has  $2^n$  vertices;
- b** the vertices can be ordered so that each one is adjacent to and has lower cost than the previous one;
- c** there exists a pivoting rule under which the simplex method requires  $2^n - 1$  changes of basis before it terminates.

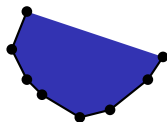
# Diameter of Polyhedra

## Definition 4.2.

- ▶ The **distance**  $d(x, y)$  between two vertices  $x, y$  is the minimum number of edges required to reach  $y$  starting from  $x$ .
- ▶ The **diameter**  $D(P)$  of polyhedron  $P$  is the maximum  $d(x, y)$  over all pairs of vertices  $(x, y)$ .
- ▶  $\Delta(n, m)$  is the maximum  $D(P)$  over **all polytopes** in  $\mathbb{R}^n$  that are represented in terms of  $m$  inequality constraints.
- ▶  $\Delta_u(n, m)$  is the maximum  $D(P)$  over **all polyhedra** in  $\mathbb{R}^n$  that are represented in terms of  $m$  inequality constraints.



$$\Delta(2, 8) = \lfloor \frac{8}{2} \rfloor = 4$$



$$\Delta_u(2, 8) = 8 - 2 = 6$$

# Hirsch Conjecture

**Observation:** The diameter of the feasible set in a linear programming problem is a lower bound on the number of steps required by the simplex method, no matter which pivoting rule is being used.

## Polynomial Hirsch Conjecture

$$\Delta(n, m) \leq \text{poly}(m, n)$$

### Remarks

- ▶ Known lower bounds:  $\Delta_u(n, m) \geq m - n + \lfloor \frac{n}{5} \rfloor$
- ▶ Known upper bounds:

$$\Delta(n, m) \leq \Delta_u(n, m) < m^{1+\log_2 n} = (2n)^{\log_2 m}$$

- ▶ The **Strong Hirsch Conjecture**

$$\Delta(n, m) \leq m - n$$

was disproven in 2010 by Paco Santos for  $n = 43$ ,  $m = 86$ .



## Average Case Behavior of the Simplex Method

- ▶ Despite the exponential lower bounds on the worst case behavior of the simplex method (Klee-Minty cubes etc.), the simplex method usually behaves well in practice.
- ▶ The number of iterations is “typically”  $O(m)$ .
- ▶ There have been several attempts to explain this phenomenon from a more theoretical point of view.
- ▶ These results say that “on average” the number of iterations is  $O(\cdot)$  (usually polynomial).
- ▶ One main difficulty is to come up with a meaningful and, at the same time, manageable definition of the term “on average”.

