Notes on games in temporal logic

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Abstract

We prove a positive and a negative expressive completeness result, both due to J.A.W. (Hans) Kamp. We use games to simplify the proofs.

1 Introduction

Temporal logic is a broad term covering various logics designed for expressing and reasoning about properties of time. First studied in philosophy, it has made great inroads into computer science, where time is an omnipresent phenomenon. While classical first-order logic offers an alternative 'extensional' formalism for working with time, temporal logic offers certain advantages, such as its arguably greater intuitivity, lower computational complexity, flexibility (various different choices of temporal connective can be made, according to need), and indeed its 'intensional' nature, whereby formulas are evaluated 'locally', e.g., at points of time, rather than globally, from 'outside' time. Temporal logics are therefore perhaps more natural to use, and lend themselves to automated execution.

To confirm claims of the various temporal logics as viable competitors of classical logic, it is important to establish how their expressive power compares with that of first-order logic. This question was first addressed by Hans Kamp, who in [10] showed that in the propositional case, over dedekind-complete linear flows of time, the temporal logic with connectives Until and Since is exactly as expressive as first-order logic. In part 2 of this paper, we prove this theorem, using a simplified approach based on that in [5]. Our proof uses rather unusual back-and-forth games, which seem to clarify the arguments, together with a quick concluding argument due to Thomas Wilke. [5] also states that two further connectives can be added to obtain a similar result without the dedekind-complete restriction. Another proof of these results, due to Gabbay, uses 'separation'; see [4].

These facts hold for propositional temporal logic, where the atomic formulas are propositional atoms which can be either true or false at each time point. In many applications, however, more expressivity is needed: we have an entire first-order structure associated with each time point, and the temporal logic appropriate for it should combine the first-order logic of these structures with temporal connectives to express variations in their properties over time. The corresponding first-order formalism is a two-sorted one, with sorts for time and for the domain of the structures (here assumed constant over time). In contrast to his positive results in the propositional case, Kamp showed in [11] that Until and Since are not as expressive as two-sorted first-order logic over rationals flow of time. A similar result was proved for natural numbers flow of time in [2, 1]. In Part 1, we prove the result for dedekind-complete time, again using games. We also present a positive result from [9], giving a natural fragment of two-sorted first-order logic which does match the expressivity of temporal logic, in the dedekind-complete case. Further positive results for 'predicate temporal logic' can be found in [9].

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Part 1: Kamp's theorem

We will prove the theorem of Kamp [11] that *first-order* temporal logic with Until and Since is not expressively expressively complete over the class of all linear (even dedekind-complete) flows of time. The proof is presented in terms of games.

2 Logic for temporal structures

Let L be a first-order signature, for simplicity without function or constant symbols. We consider structures of the following form.

DEFINITION 2.1 (TEMPORAL STRUCTURES) Temporal first-order structures with signature L have the form $\mathcal{M} = (T, D_t)_{t \in T}$, where $T = (T, <)$ is an irreflexive linear order (the flow of time) and for each $t \in T$, D_t is an L-structure, such that all the D_t have the same domain, say D.

There are two appropriate logics for such structures. Our aim is to compare their expressive power.

2.1 Temporal logic

DEFINITION 2.2 (FIRST-ORDER TEMPORAL LOGIC) First-order temporal (L-)formulas are made as follows:

1. Any atomic L-formula is one. E.g., $R(\bar{x})$, $x = y$.
2. If A, B are formulas then so are $\neg A$, $A \wedge B$, $\exists x A$, $U(A, B)$, and $S(A, B)$.

DEFINITION 2.3 (SEMANTICS) For a temporal first-order L-structure $\mathcal{M} = (T, D_t)_{t \in T}$ with domain D, an assignment h of its free variables to elements of D, a time $t \in T$, and a formula A, we define $\mathcal{M}, h, t \models A$ by induction on A:

1. If A is atomic then $\mathcal{M}, h, t \models A$ iff $D_t, h \models A$.
2. $\wedge, \neg$: as usual.
3. $\mathcal{M}, h, t \models \exists x A$ iff $\mathcal{M}, h', t \models A$ for some h' with $h'(y) = h(y)$ for all variables $y \neq x$.
4. $\mathcal{M}, h, t \models U(A, B)$ iff there exists $u \in T$ with $u > t$, $\mathcal{M}, h, u \models A$, and $\mathcal{M}, h, v \models B$ for all $v \in T$ with $t < v < u$.
5. $\mathcal{M}, h, t \models S(A, B)$ iff there exists $u \in T$ with $u < t$, $\mathcal{M}, h, u \models A$, and $\mathcal{M}, h, v \models B$ for all $v \in T$ with $u < v < t$.

So we read $U(A, B)$ as 'B will be true *until* A is', and $S(A, B)$ as '*since* A was true, B has been'.

2.2 Two-sorted first-order logic

DEFINITION 2.4 (2-SORTED LOGIC) An alternative logic for temporal structures is 2-sorted first-order logic, with first sort the common domain D of the D_t, and second sort the linear ordering T. The idea is to index the predicates $R \in L$ by the time t. The signature is

$$L' = \{t < t\} \cup \{R'(t, \bar{x}) : R(\bar{x}) \in L\},$$

where $\bar{x}$ denotes sort D and t denotes sort T. We will write x, y, z for domain variables, and u, v, w for time variables.

An important fragment, discovered by Wolter and Zakharyaschev and called (here) the *simple fragment,* consists of all formulas such that any subformula of the form $\exists x \varphi$ has at most one free time variable.

We view a temporal structure $\mathcal{M} = (T, D_t)_{t \in T}$ as a 2-sorted structure with sorts T, D, as follows. Pick an assignment h of D-variables into the common domain D, and an assignment τ of T-variables into T. For atomic formulas $R'(u, \bar{x})$, we define $\mathcal{M}, h, \tau \models R'(u, \bar{x})$ iff $D_{\tau(u)}, h \models R(\bar{x})$. We now define $\mathcal{M}, h, \tau \models \varphi$ for arbitrary 2-sorted φ in the usual first-order way.

3 Comparing them

DEFINITION 3.1 (EQUIVALENCE) Let $A(\bar{x})$ be a first-order temporal formula and $\varphi(u, \bar{x})$ a 2-sorted formula with at most one free time variable, say u. We say that A and φ are *equivalent* if for any structure $\mathcal{M} = (T, D_t)_{t \in T}$, assignment h of domain variables into the common domain D, and assignment τ mapping variable u into T, we have

$$\mathcal{M}, h, \tau \models \varphi(u, \bar{x}) \iff \mathcal{M}, h, \tau(u) \models A(\bar{x}).$$

We say that A and φ are *equivalent over dedekind-complete time* if the above holds whenever T is dedekind-complete.

DEDEKIND COMPLETENESS Recall that a linear order $(T, <)$ is *dedekind complete* if, for every non-empty subset S of T, if S has a lower bound in T then it has a greatest lower bound, written $\inf(S)$, and if S has an upper bound in T then it has a least upper bound, written $\sup(S)$. Formally, $\inf(S) \in T$, $\inf(S) \leq s$ for all $s \in S$, and if $t \in T$ and $t \leq s$ for all $s \in S$ then $t \leq \inf(S)$; a similar definition can be given for $\sup(S)$. Examples of dedekind complete orders are $\mathbb{N}$ (the natural numbers), $\mathbb{Z}$ (integers), $\mathbb{R}$ (reals), and any non-zero ordinal. Non-examples are $\mathbb{Z} + \mathbb{Z}$ (a copy of $\mathbb{Z}$ followed by another), and $\mathbb{Q}$ (the rationals).

DEFINITION 3.2 (STANDARD TRANSLATION) Any temporal formula $A(\bar{x})$ has a *standard translation,* a 2-sorted first-order formula $A'(u, \bar{x})$, defined inductively:

1. $R(\bar{x})' = R'(u, \bar{x})$
2. $(x = y)' = (x = y)$
3. $(A \wedge B)' = A' \wedge B'$, and $(\neg A)' = \neg A'$
4. $(\exists x A)' = \exists x(A')$
5. $U(A, B)'(u, \bar{x}) = \exists v > u(A'(v, \bar{x}) \wedge \forall w(u < w < v \to B'(w, \bar{x})))$
6. $S(A, B)'(u, \bar{x}) = \exists v < u(A'(v, \bar{x}) \wedge \forall w(v < w < u \to B'(w, \bar{x})))$.

From the definition, we obtain

PROPOSITION 3.3 *For any temporal formula A, A' is simple and equivalent to A.*

Is there a converse translation, even in principle (without knowing how to describe it)? Formally:

DEFINITION 3.4 (EXPRESSIVE COMPLETENESS) We say that first-order temporal logic as in definition 2.2 is *expressively complete (over dedekind-complete time, resp.)* if for every 2-sorted formula $\varphi(u, \bar{x})$ with at most one free time variable, there is a temporal formula $A(\bar{x})$ that is equivalent to φ (resp. over dedekind-complete time).

Is first-order temporal logic expressively complete (over dedekind-complete time)? The answer is 'no' (Kamp; see theorem 5.1). But there is a positive result for simple formulas (next section).

4 Expressive completeness holds for simple formulas

THEOREM 4.1 ([9]) *Let $\varphi(u, \bar{x})$ be a* simple *(2-sorted) formula. Then there exists a temporal formula $\varphi^*(\bar{x})$ that is equivalent to φ over dedekind-complete time.*

PROOF:

The proof is by induction on the number of domain quantifiers $\exists x$ (for any domain variable x) in φ. Let Ψ be the set of all simple 2-sorted formulas with at most one free time variable and fewer domain quantifiers than φ, and inductively assume the result for all formulas in Ψ. Then φ is built from atomic formulas of the form $P(v, \bar{x})$, $v < w$, $v = w$, $x = y$, and formulas of the form $\exists x \psi(v, x, \bar{y})$ for $\psi \in \Psi$, using only the boolean operations and quantifiers over time variables. For each such ψ, introduce a unary relation symbol Q_ψ and replace the subformula $\exists x \psi(v, x, \bar{y})$ in φ by $Q_\psi(v)$. Similarly, for each atomic subformula α of φ of the form $P(v, \bar{x})$ or $x = y$, introduce a new unary relation symbol Q_α and replace α in φ by $Q_\alpha(v)$ (in the case $x = y$, v is arbitrary). In this way, we obtain a first-order formula $\varphi'(u)$ involving only time variables, monadic predicates, equality, and $<$. By Kamp's theorem [10], proved here in theorem 6.1, there is a propositional US-formula $\varphi^\dagger$, whose atoms are q_ψ, q_α for ψ, α as above, that is equivalent to φ' over dedekind-complete time.

By the inductive hypothesis, for all $\psi(v, x, \bar{y}) \in \Psi$ there is a temporal formula $\psi^*(x, \bar{y})$ equivalent to ψ. Hence, $\exists x \psi(v, x, \bar{y})$ is equivalent to $\exists x \psi^*(x, \bar{y})$. Now, in $\varphi^\dagger$, replace the atoms q_ψ by $\exists x \psi^*(x, \bar{y})$, for all such ψ, and replace the q_α for atomic α of the form $P(v, \bar{x})$ by an atomic formula $p(\bar{x})$, say, whose standard translation with free time variable v is $P(v, \bar{x})$. Replace subformulas $q_{x=y}$ of $\varphi^\dagger$ by $x = y$. We obtain a temporal formula φ^* that is clearly equivalent to φ. $\square$

We will see in the next section that the requirement of simplicity is necessary.

5 Expressive completeness fails in general

Let L consist of a single unary relation symbol (say, $P(x)$). So $L' = \{P'(t, x), t < t\}$. Kamp proved in [11] that

$$\alpha = \exists uv\big(u < v \wedge \forall x(P'(u, x) \leftrightarrow P'(v, x))\big)$$

cannot be expressed in temporal logic over the flow of time $(\mathbb{Q}, <)$.[1] This is in a way no surprise at all: α is not simple. We prove the following slight strengthening of Kamp's result:

THEOREM 5.1 *First-order temporal logic is not expressively complete, even over dedekind-complete time.*

PROOF:

We show using Ehrenfeucht–Fraïssé games that α is not equivalent to any simple (2-sorted) sentence over dedekind-complete time. Because by proposition 3.3, any temporal sentence is equivalent over dedekind-complete time to a simple sentence, first-order temporal logic is not expressively complete over dedekind-complete (or linear) time.

We define temporal L-structures $\mathcal{M} = (\mathbb{R}, D_t)_{t \in \mathbb{R}}$ and $\mathcal{M}' = (\mathbb{R}, D'_t)_{t \in \mathbb{R}}$, as follows. We define the domains of D_t, D'_t (all $t \in \mathbb{R}$) to be $\mathbb{N}$. Let

$$\mathcal{P} = \{(X, Y) : X, Y \subseteq \mathbb{N},\ X \cap Y = \emptyset,\ |X \cup Y| \leq k\}.$$

As $\mathcal{P}$ is countable, we may partition $\mathbb{R}$ into dense subsets $Q_\pi \subseteq \mathbb{R}$, for $\pi \in \mathcal{P}$. Also, by Sierpiński's lemma [7, exercise 4.2.4], we may choose sets $S_t \subseteq \mathbb{N}$ for $t \in \mathbb{R}$ such that if $t \neq t'$ then $S_t \setminus S_{t'}$ is infinite. Let S'_t be the set of even natural numbers, for all $t \in \mathbb{R}$. Arrange that for all $t \in \mathbb{R}$ and $d \in D$,

$$D_t \models P(d) \iff \forall (X, Y) \in \mathcal{P}\big(t \in Q_{(X,Y)} \to d \in (S_t \cup X) \setminus Y\big),$$

[1] A similar English sentence: 'Everyone who gets married today will get divorced on the same day'. An analogous result was proved for the flow of time $(\mathbb{N}, <)$, the natural numbers, in [2, 1]. Question: can games be used to prove this?

and a similar clause for D'_t.

Now define a k-pebble Ehrenfeucht-Fraïssé back-and-forth game between $\mathcal{M}, \mathcal{M}'$, viewed as 2-sorted L'-structures. The game has length ω, but there is a restriction, matching the definition of 'simple': whenever $\forall$ moves or adds a domain pebble, there must be at most one time pebble in play (we mean one pair, with a pebble of the pair on each structure). Standard game arguments now show that (γ) if $\exists$ has a winning strategy in the k-pebble game then $\mathcal{M}, \mathcal{M}'$ agree on all simple sentences using k variables. Cf. [3].

We show that $\exists$ has a winning strategy in this game. Her strategy is simply to ensure that the pebbles' positions always define a partial isomorphism from $\mathcal{M}$ to $\mathcal{M}'$.

- Clearly, for each $t \in \mathbb{R}$, infinitely many points in D_t and D'_t satisfy P and infinitely many points satisfy $\neg P$. It follows that any finite partial isomorphism : $D_t \to D'_{t'}$ extends to an isomorphism. This can be used by $\exists$ to respond to any domain move made by $\forall$, preserving the partial-isomorphism condition.
- If $\forall$ makes a time move, moving a pebble to $t \in \mathbb{R}$ in $\mathcal{M}$, say, then $\exists$ defines X (respectively, Y) to be the set of those $d' \in D'$ currently pebbled in $\mathcal{M}'$ such that the corresponding element d of D in $\mathcal{M}$ satisfies $D_t \models P(d)$ (respectively, $D_t \models \neg P(d)$). As $Q_{(X,Y)}$ is dense in $\mathbb{R}$, $\exists$ can choose $t' \in Q_{(X,Y)}$ in the appropriate position among the currently-pebbled time points to preserve the partial isomorphismness on the time sort. Then, the pebbled elements in $D_t, D'_{t'}$ have the same isomorphism type, so $\exists$ has kept to her strategy.

So $\exists$ has a winning strategy in the game between $\mathcal{M}, \mathcal{M}'$, and it follows by ($\gamma$) that these structures must agree on all k-variable simple sentences. But clearly, $\mathcal{M} \models \neg\alpha$, while $\mathcal{M}' \models \alpha$ because for any $\pi \in \mathcal{P}$, if $t, t' \in Q_\pi$ then $D'_t = D'_{t'}$. Since this construction can be undertaken for any finite k, and $(\mathbb{R}, <)$ is dedekind-complete, we conclude that α is not equivalent to any simple sentence over dedekind-complete time. □

Question: are there larger natural fragments than the simple formulas with respect to which temporal logic remains expressively complete (over dedekind-complete time)?

Part 2: Kamp's theorem

We will prove the theorem of Kamp [10] that *propositional* temporal logic with Until and Since is expressively complete over the class of all dedekind complete flows of time. We will use the argument of Gabbay, Pnueli, Shelah, and Stavi [5], but presented in terms of games. Section 9 uses a method of Thomas Wilke.

6 Two logics for $\mathcal{D}$

We fix an arbitrary *finite* set L of propositional atoms. These represent time-dependent facts. We write $\mathcal{D}$ for the class of all triples $(T, <, h)$, where $(T, <)$ is a non-empty irreflexive dedekind complete linear order, and $h : L \to \wp(T)$ is a map, the 'assignment map'. We will use M, N to denote structures in $\mathcal{D}$. If $M = (T, <, h) \in \mathcal{D}$, we will habitually abuse notation by identifying M with T. So, for example, we write $a \in M$ to mean that a is an element of T, and $\bar{a} \in M$ to mean that $\bar{a}$ is a tuple of elements of T. We write $|\bar{a}|$ for the length of the tuple $\bar{a}$.

We define two logics to talk about structures in $\mathcal{D}$.

6.1 Monadic first-order logic

For each atom $p \in L$ we associate a unary relation symbol P. The monadic formulas are then defined as follows:

- $P(x)$, $x = y$, and $x < y$ are atomic formulas, for any $p \in L$ and any variables x, y;
- if φ and ψ are monadic formulas, then so are $\neg\varphi$, $\varphi \wedge \psi$, and $\exists x\varphi$ (for any variable x).

Semantics: if $t \in M = (T, <, h) \in \mathcal{D}$, we let $M \models P(t)$ iff $t \in h(p)$, for any atom $p \in L$. The other clauses are given semantics as usual. We will use φ, ψ, χ, etc. (Greek) to denote monadic formulas.

6.2 Temporal logic

- Any atom $p \in L$ is a temporal formula, of rank 0.
- $\top$ is a temporal formula, also of rank 0.
- If A, B are temporal formulas of ranks r, s, respectively, then
  - $\neg A$ is a temporal formula, of rank r;
  - $A \wedge B$ is a temporal formula, of rank $\max(r, s)$;
  - $U(A, B)$, $S(A, B)$ are temporal formulas, of rank $1 + \max(r, s)$.

So the rank of a temporal formula A is the maximum depth of nesting of temporal connectives in A. For example, the formula $\neg U(p \wedge S(q, \neg r), \neg U(p, r))$ has rank 2.

Semantics: let $M = (T, <, h) \in \mathcal{D}$ and $t \in M$. Then:

- $M, t \models p$ iff $t \in h(p)$, for all $p \in L$;
- $M, t \models \top$;
- $M, t \models \neg A$ iff $M, t \not\models A$;
- $M, t \models A \wedge B$ iff $M, t \models A$ and $M, t \models B$;
- $M, t \models U(A, B)$ iff there is $u > t$ in T such that $M, u \models A$, and for all $v \in T$ with $t < v < u$ we have $M, v \models B$;
- $M, t \models S(A, B)$ iff there is $u < t$ in T such that $M, u \models A$, and for all $v \in T$ with $u < v < t$ we have $M, v \models B$.

6.3 The main theorem

We will prove

THEOREM 6.1 [KAMP, [10]] *For any finite L, and any monadic formula $\varphi(x)$ with one free variable, x, there is a temporal formula A such that for all $M \in \mathcal{D}$ and all $a \in M$,*

$$M, a \models A \iff M \models \varphi(a).$$

That is, over dedekind complete time, any monadic formula has an equivalent temporal formula written with Until and Since. We thus say that the connectives Until and Since are *expressively complete* over dedekind complete time. Note that they are not always expressively complete over non-dedekind complete time, such as $\mathbb{Q}$.

6.4 Notation

Definition 6.2 (Types) Let $M \in \mathcal{D}$, $a \in M$, and $r < \omega$. We write $tp_r^M(a)$ for

$$\bigwedge\{A : A \text{ a temporal formula of rank at most } r,\ M, a \models A\}.$$

If $S \subseteq M$, we write $tp_r^M(S)$ for $\bigvee\{tp_r^M(a) : a \in S\}$.

Remark 1 As L is finite, it can be seen by induction on r that there are only finitely many logically inequivalent temporal formulas of rank r. Hence we can take $tp_r^M(a)$, $tp_r^M(S)$ to be temporal formulas of rank r.

Clearly, two points have the same r-type iff they satisfy the same temporal formulas of rank at most r, and an element x satisfies $tp_r^M(S)$ iff it has the same r-type as some element of S.

Definition 6.3 (Intervals) We use the standard notation (x, y) for the interval $\{z : x < z < y\}$ of a linear order $(T, <)$, where $x < y$ in T. For intervals I, J, we write $I < J$ if for all $x \in I$, $y \in J$, we have $x < y$. If $\bar{a}$ is a tuple of elements of $(T, <)$, an *interval generated by* $\bar{a}$ is a maximal convex subset of T that is disjoint from $\bar{a}$. Thus, for example, if $\bar{a}$ is $a_1, \ldots, a_n$, and $a_1 < \cdots < a_n$, the intervals $\bar{a}$ generates are $(-\infty, a_1)$, (a_1, a_2), $\ldots$, (a_{n-1}, a_n), (a_n, ∞). Here, and below, we let $(-\infty, a)$ denote $\{x \in T : x < a\}$, and (a, ∞) denote $\{x \in T : x > a\}$. If $\bar{b}$ is a tuple of the same order type as $\bar{a}$, and I an interval generated by $\bar{a}$, then *the interval generated by* $\bar{b}$ *corresponding to* I is the one whose endpoints are corresponding elements of $\bar{b}$. For example, with $\bar{a}$ as above, we would have $\bar{b} = b_1, \ldots, b_n$ with $b_1 < \cdots < b_n$, and the interval generated by $\bar{b}$ corresponding to (a_4, a_5) would be (b_4, b_5), etc.

7 Short games

Definition 7.1 (The short game) Let $M, N \in \mathcal{D}$, and let $\bar{a} \in M$, $\bar{a}' \in N$ be tuples of equal length. Let $n, r < \omega$. We define a rather short game

$$G_{n;r}(M, \bar{a}; N, \bar{a}')$$

as follows. There are two players, $\forall$ (male), and $\exists$ (female).[2] The game has only two rounds. In the first round, $\forall$ chooses n elements $b_1, \ldots, b_n$ from M, not necessarily distinct. $\exists$ responds by choosing elements $b'_1, \ldots, b'_n$ of N. We write $\bar{b}$ for the tuple $b_1, \ldots, b_n$, and $\bar{b}'$ for $b'_1, \ldots, b'_n$. In the second round, $\forall$ chooses a single element c' of N, and $\exists$ responds with an element $c \in M$. This completes play. $\exists$ wins iff the following hold:

1. The two concatenated tuples $\bar{a}\bar{b}c$ and $\bar{a}'\bar{b}'c'$ have the same order type (i.e., they satisfy the same quantifier-free first-order formulas written with $=, <$);
2. if $x \in M$ is an element of $\bar{a}\bar{b}c$, and $x' \in N$ is the corresponding element of $\bar{a}'\bar{b}'c'$, then for every temporal formula A of rank at most r, we have $M, x \models A$ iff $N, x' \models A$. That is, $tp_r^M(x) = tp_r^N(x')$.

Evidently, if $\exists$ has a winning strategy for a game $G_{n;r}$ then she also has one for $G_{n';r'}$ for all $n' \leq n$ and $r' \leq r$. We will use this fact often, without saying so explicitly. So, for example, in $G_{n;r}(M, \bar{a}; N, \bar{a}')$ we permit $\forall$ to pick fewer than n elements of M in round 1, if he wishes. The following is also useful.

Lemma 7.2 *Fix a short game $G_{n;r}(M, \bar{a}; N, \bar{a}')$. The following are equivalent:*

1. *$\exists$ has a winning strategy for $G_{n;r}(M, \bar{a}; N, \bar{a}')$;*

[2] This convention is Hodges'.

2. *$\exists$ can win $G_{n;r}(M,\bar{a};N,\bar{a}')$ whenever $\forall$'s choice in round 1 consists of distinct elements of a single interval generated by $\bar{a}$, and his choice in round 2 is an element of the corresponding interval generated by $\bar{a}'$ (or an element of $\bar{a}'$).*

PROOF:

Only $2 \Rightarrow 1$ requires proof, and this is an easy consequence of simple facts about linear order. See [8] for details. $\square$

8 Symmetry of the short game

The reader will have noticed the asymmetry of the 'forward' and 'backward' directions of the game. We show now that this is not as bad as it looks.

DEFINITION 8.1 Let $f : \omega \to \omega$ be a function such that

- $f(0) \geq 1$, and
- $f(n+1) \geq f(n) + 2$.

For example, $f(n) = 2n + 1$.

LEMMA 8.2 *Suppose that $M, N \in \mathcal{D}$, and $\bar{a} \in M$, $\bar{a}' \in N$ are tuples of equal length. Then for all $n, r < \omega$, if $\exists$ has a winning strategy for $G_{f(n);n+r}(M,\bar{a};N,\bar{a}')$, then $\exists$ has a winning strategy for $G_{n;r}(N,\bar{a}';M,\bar{a})$.*

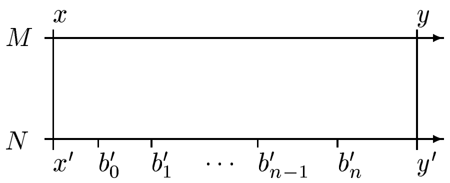
This says that if $\exists$ has winning strategies for enough 'forward' games $G_{n;r}(M,\bar{a};N,\bar{a}')$ then she has a winning strategy for a given 'backward' game $G_{n;r}(N,\bar{a}';M,\bar{a})$. It is really a syntactic result — we could equally prove that a certain class of formulas is closed under negation up to equivalence, which is what is suggested, without proof, in [5]. However, the approach via games seems rather simpler to present.

PROOF:

We will prove the lemma by induction on n. The case $n = 0$ is clear, for all $r, \bar{a}, \bar{a}'$, as the first round of $G_{0;r}(N,\bar{a}';M,\bar{a})$ is essentially empty, so the game looks like the first round of $G_{1;r}(M,\bar{a};N,\bar{a}')$, in which $\exists$ has a winning strategy. Assume that the lemma holds for n, and that $\exists$ has a winning strategy in $G_{f(n+1);n+1+r}(M,\bar{a};N,\bar{a}')$. (So the order types of $\bar{a}, \bar{a}'$ are the same, and corresponding elements have the same r-type — or she would already have lost.) We prove that she has a winning strategy in $G_{n+1;r}(N,\bar{a}';M,\bar{a})$.

By lemma 7.2, we can safely assume that in round 1 of $G_{n+1;r}(N,\bar{a}';M,\bar{a})$, $\forall$ played distinct elements of a single interval, (x', y'), say, generated by $\bar{a}'$, and that in round 2 he will pick an element of (x, y), the corresponding interval of M generated by $\bar{a}$, or of $\bar{a}$ itself. If in round 1 he played at most n elements, then $\exists$ can win, because she has a winning strategy for $G_{f(n);n+r}(M,\bar{a};N,\bar{a}')$, and so, by the inductive hypothesis, one for $G_{n;r}(N,\bar{a}';M,\bar{a})$, too. It suffices to apply this strategy.

So from now on, we will suppose that $\forall$ played elements $b'_0, \ldots, b'_n \in N$, with $x' < b'_0 < \cdots < b'_n < y'$.



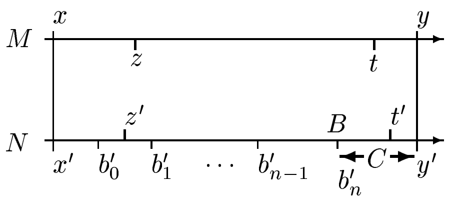
$\exists$ begins by defining the following rank r formulas (see definition 6.2):

- $B = tp_r^N(b'_n)$;
- $C = tp_r^N((b'_n, y'))$.

Then she defines $z', t' \in N$ by:

- $z' = \sup\{v' \in (x', y') : N, v' \models \neg C\}$;
- $t' = \sup\{v' \in (x', y') : N, v' \models B\}$.

Here, she lets $\sup(\emptyset) = x'$, if required. She defines $z, t \in M$ similarly — i.e., $t = \sup\{v \in (x, y) : M, v \models B\}$, etc. *These elements exist, by dedekind completeness of (the orders of) M, N;* this is the only place that the assumption of dedekind completeness is used. Notice that $x' \leq z' \leq b'_n \leq t' \leq y'$ in N. The claim below shows that $\bar{a}zt$ and $\bar{a}'z't'$ have the same order type, so that $z \leq t$, too.



Claim: $\exists$ has a winning strategy in the forward game $G_{f(n);n+r+1}(M, \bar{a}zt; N, \bar{a}'z't')$.

Proof of claim: $\exists$ will play a 'private' game to help her win the 'main' or 'real' game. The private game will be $G_{f(n+1);n+1+r}(M, \bar{a}; N, \bar{a}')$, in which $\exists$ has a winning strategy. The 'real' game is, of course, $G_{f(n);n+r+1}(M, \bar{a}zt; N, \bar{a}'z't')$.

In the first round of $G_{f(n);n+r+1}(M, \bar{a}zt; N, \bar{a}'z't')$, $\forall$ chooses a tuple $\bar{s}$ of elements of M. By lemma 7.2, we can assume that $\bar{s}$ consists of at most $f(n)$ distinct points in some interval generated by $\bar{a}zt$. $\exists$ privately adds z and t to $\forall$'s choices, making a tuple $\bar{s}zt$ involving at most $f(n) + 2$ distinct points from any interval generated by $\bar{a}$. Now $f(n)+2 \leq f(n+1)$, so $\bar{s}zt$ is a valid move for $\forall$ in the private game $G_{f(n+1);n+1+r}(M, \bar{a}; N, \bar{a}')$, and $\exists$ can apply her winning strategy. This delivers a tuple $\bar{s}^*z^*t^*$ of elements of N, and $\exists$ responds to $\forall$ with $\bar{s}^*$ in the real game.

$\forall$ will then continue the real game by choosing a single point u^* of N. $\exists$ simply pretends that $\forall$ continued the private game with u^*. Her strategy delivers some point $u \in M$ in response, and it is u that $\exists$ plays in response to u^* in the real game.

Let us check that $\exists$ wins the real game, $G_{f(n);n+r+1}(M, \bar{a}zt; N, \bar{a}'z't')$. We require that the tuples $\bar{a}zt\bar{s}u$ and $\bar{a}'z't'\bar{s}^*u^*$ have the same order type, and corresponding elements satisfy the same temporal formulas of rank at most $n + r + 1$. Her strategy wins $G_{f(n+1);n+1+r}(M, \bar{a}; N, \bar{a}')$, so the tuples $\bar{a}\bar{s}ztu$ and $\bar{a}'\bar{s}^*z^*t^*u^*$ have the same order type, and corresponding elements satisfy the same temporal formulas of rank at most $n + 1 + r$. Obviously, the same holds for the tuples $\bar{a}zt\bar{s}u$ and $\bar{a}'z^*t^*\bar{s}^*u^*$. So it only remains to check that the responses z^*, t^* of her strategy to z and t are z' and t', respectively. If we can prove this, we will have established the claim.

Consider z^*. First, note that $z^* \leq z'$. For if $z = x$, then as the strategy preserves order types, $z^* = x' \leq z'$. If $z \in (x, y)$, then $M, z \models \neg C \vee \neg S(\top, C)$. As this formula has rank $r + 1$, and the strategy preserves rank $n + 1 + r$ temporal formulas, we have $N, z^* \models \neg C \vee \neg S(\top, C)$. So by definition of z', we see that $z^* \leq z'$. And if $z = y$ (actually impossible!), then $M, z \models \neg S(\top, C)$, so $N, z^* \models \neg S(\top, C)$, and again we obtain $z^* \leq z'$.

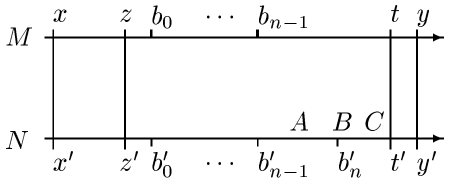
If $z^* < z'$, then there is some v' with $N, v' \models \neg C$ and $z^* < v' < y'$. If $\forall$ were to play such a v' in the second round of $G_{f(n+1);n+1+r}(M, \bar{a}; N, \bar{a}')$, $\exists$ would have no winning response in M, as there is no $v \in M$ with $M, v \models \neg C$ and $z < v < y$. This contradicts the assumption that the strategy wins. So, indeed, $z^* = z'$. The argument for t^* is similar. This proves the claim.

So, by the claim and the inductive hypothesis, $\exists$ has a winning strategy in the n-backward short game $G_{n;r+1}(N, \bar{a}'z't'; M, \bar{a}zt)$. She adopts this as her new private game, to help her win the main game $G_{n+1;r}(N, \bar{a}'; M, \bar{a})$.

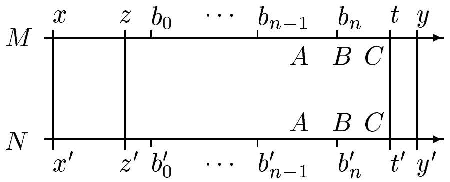
The first possibility is that $b'_0, \ldots, b'_n$ were split up by the new points z', t', so that at most n of them lie in any one subinterval of (x', y') generated by $\bar{a}'z't'$. In this case, $b'_0, \ldots, b'_n$ is a valid move of the private game $G_{n;r+1}(N, \bar{a}'z't'; M, \bar{a}zt)$, so $\exists$ may apply her winning strategy. Clearly, this will win $G_{n+1;r}(N, \bar{a}'; M, \bar{a})$ for her, and we are done.

So suppose otherwise. Then as $z' \leq b'_n \leq t'$, we must have $b'_0, \ldots, b'_n \in (z', t')$. In this case, $\exists$ first uses her strategy in $G_{n;r+1}(N, \bar{a}'z't'; M, \bar{a}zt)$ to respond to the first n points, $b'_0, \ldots, b'_{n-1}$, only. This yields points $b_0, \ldots, b_{n-1}$ in (z, t), satisfying the same rank $r+1$-formulas as $b'_0, \ldots, b'_{n-1}$.

Define $b_{-1} = z$, $b'_{-1} = z'$ (in case $n = 0$), and let $A = tp_r^N((b'_{n-1}, b'_n))$.



Now $N, b'_{n-1} \models U(B, A)$. As $U(B, A)$ has rank $r+1$, so is preserved, we have $M, b_{n-1} \models U(B, A)$. But $M, t \models B \vee \neg S(\top, \neg B))$ if $t < y$, and $M, t \models \neg S(\top, \neg B)$ if $t = y$, so we can witness the B in '$M, b_{n-1} \models U(B, A)$' by some b_n with $b_{n-1} < b_n < y$. Player $\exists$ adds such a b_n to her choices $b_0, \ldots, b_{n-1}$, and responds to $\forall$ in round 1 of the main game $G_{n+1;r}(N, \bar{a}'; M, \bar{a})$ with $b_0, \ldots, b_n$.



We move to round 2. Recall that $\forall$ now chooses some c in (x, y) or in $\bar{a}$. $\exists$ can use her strategy in $G_{n;r+1}(N, \bar{a}'z't'; M, \bar{a}zt)$ to respond, except when $b_{n-1} < c < y$. Here, there are three cases:

Case 1: $b_{n-1} < c < b_n$. Then, by choice of b_n, we have $M, c \models A$. By definition of A, we have $M, c \models X$, where $X = tp_r^N(c')$ for some $c' \in (b'_{n-1}, b'_n)$. $\exists$ then responds with such a c'.

Case 2: $c = b_n$. $\exists$ then responds with b'_n.

Case 3: $b_n < c < y$. Recall that $z \leq b_{n-1} \leq b_n$, even when $n = 0$. So $z < c$. So, by definition of z, $M, c \models C$. That is, $M, c \models Y$, where $Y = tp_r^N(c')$ for some $c' \in (b'_n, y')$. $\exists$ then responds with any such c'.

It can be seen that, in all cases, the order types of $\bar{a}\bar{b}c$ and $\bar{a}'\bar{b}'c'$ are the same, and that corresponding elements of these tuples satisfy the same temporal formulas of rank at most r (they have the same r-type). Thus, $\exists$ has won. This proves the main lemma. $\square$

9 Finishing off

The following approach is due to T. Wilke (personal communication, 1997).

Definition 9.1 For $n, r < \omega$, let L_r be L plus a unary relation symbol for each temporal L-formula of rank $\leq r$. We use the same notation for a temporal formula as for the corresponding relation symbol. Any linear L-structure M expands canonically to an L_r-structure by interpreting the unary relation symbol A (for a temporal formula A) as $\{t \in M : M, t \models A\}$.

An *n, r-decomposition formula* is a formula of the form

$$\beta(\bar{z}) = \exists \bar{x}\, \forall y\, \chi(\bar{x}, y, \bar{z})$$

where $\bar{x} = (x_1, \ldots, x_n)$ is an n-tuple of variables and χ is a quantifier-free L_r-formula. We write $\Delta_{n,r}$ for the set of all positive boolean combinations of n, r-decomposition formulas.

Definition 9.2 If $k < \omega$, let $\mathcal{C}_k$ be the class of all structures of the form $(M, \bar{a})$, where $M \in \mathcal{D}$ and $\bar{a}$ is a k-tuple of elements of M.

For $k, n, r < \omega$, we say that a class $\mathcal{K} \subseteq \mathcal{C}_k$ is *n, r-definable* if there is $\delta(\bar{z}) \in \Delta_{n,r}$ with $|\bar{z}| = k$, such that for all structures $(M, \bar{a}) \in \mathcal{C}_k$ we have $(M, \bar{a}) \in \mathcal{K}$ iff $M \models \delta(\bar{a})$. We say that δ *defines* $\mathcal{K}$.

Lemma 9.3 *Let $k, n, r < \omega$, and let $\mathcal{K} \subseteq \mathcal{C}_k$. The following are equivalent:*

1. *$\mathcal{K}$ is n, r-definable,*
2. *$\forall$ has a winning strategy in the short game $G_{n;r}(M, \bar{a}; N, \bar{a}')$ whenever $(M, \bar{a}) \in \mathcal{K}$ and $(N, \bar{a}') \notin \mathcal{K}$.*

Proof:

Suppose first that $\delta(\bar{z}) \in \Delta_{n,r}$ defines $\mathcal{K}$. We can assume that δ has the form

$$\bigvee_i \bigwedge_j \exists \bar{x} \forall y \chi_{ij}(\bar{x}, y, \bar{z})$$

where the χ_{ij} are as in definition 9.1. If $(M, \bar{a}) \in \mathcal{K}$ and $(N, \bar{a}') \notin \mathcal{K}$, then for some i, $M \models \bigwedge_j \exists \bar{x} \forall y \chi_{ij}(\bar{x}, y, \bar{a})$, and $N \models \neg \bigwedge_j \exists \bar{x} \forall y \chi_{ij}(\bar{x}, y, \bar{a}')$. So for some j, $N \models \neg \exists \bar{x} \forall y \chi_{ij}(\bar{x}, y, \bar{a}')$, while $M \models \exists \bar{x} \forall y \chi_{ij}(\bar{x}, y, \bar{a})$. So $M \models \forall y \chi_{ij}(\bar{b}, y, \bar{a})$ for some n-tuple $\bar{b} \in M$. Let $\forall$ choose such a $\bar{b}$ as his first-round move in $G_{n;r}(M, \bar{a}; N, \bar{a}')$. Let $\exists$ respond with $\bar{b}'$, say. Then we have $N \models \neg \forall y \chi_{ij}(\bar{b}', y, \bar{a}')$. So $\forall$ can find $c' \in N$ with $N \models \neg \chi_{ij}(\bar{b}', c', \bar{a}')$. Let him play such a c', and let $\exists$ respond with $c \in M$. Then $M \models \chi_{ij}(\bar{b}, c, \bar{a})$. Since χ_{ij} is quantifier-free with its temporal formulas of rank $\leq r$, this means that $\exists$ lost $G_{n;r}(M, \bar{a}; N, \bar{a}')$. We have given a winning strategy for $\forall$.

Conversely, first note that if $(M, \bar{a}), (N, \bar{a}') \in \mathcal{C}_k$ and $\forall$ has a winning strategy in $G_{n;r}(M, \bar{a}; N, \bar{a}')$, then there is $\delta(\bar{z}) \in \Delta_{n,r}$ such that $M \models \delta(\bar{a})$ and $N \models \neg \delta(\bar{a}')$. For, let $\forall$ begin play, using his winning strategy, by choosing the n-tuple $\bar{b} \in M$ and let $\exists$ respond with $\bar{b}' \in N$. $\forall$ then picks $c' \in N$. Now, whatever $c \in M$ $\exists$ picks, $\bar{a}\bar{b}c$ does not satisfy the same quantifier-free L_r-formulas as $\bar{a}'\bar{b}'c'$. Let $\chi_{\bar{b}'}(\bar{x}, y, \bar{z})$ be a quantifier-free L_r-formula defining precisely the quantifier-free L_r-formulas that $\bar{b}'c'\bar{a}'$ satisfies. There is such a formula, and it depends only on $\bar{b}'$, since c' is determined by $\bar{b}'$ and $\forall$'s strategy. Then

$$M \models \forall y \neg \chi_{\bar{b}'}(\bar{b}, y, \bar{a}) \text{ for all } n\text{-tuples } \bar{b}' \in N, \tag{1}$$

while $N \models \chi_{\bar{b}'}(\bar{b}', c', \bar{a}')$ and so

$$N \models \exists y \chi_{\bar{b}'}(\bar{b}', y, \bar{a}') \text{ for all } n\text{-tuples } \bar{b}' \in N. \tag{2}$$

Now let

$$\delta(\bar{z}) = \exists \bar{x} \forall y \bigwedge_{\bar{b}' \in N} \neg \chi_{\bar{b}'}(\bar{x}, y, \bar{z}).$$

The conjunction here is essentially a finite one, because there are up to logical equivalence only finitely many quantifier-free L_r-formulas with $k+n+1$ free variables, so δ is an n,r-decomposition formula. We claim that $M \models \delta(\bar{a})$ and $N \models \neg\delta(\bar{a}')$. For $M \models \delta(\bar{a})$, note that $M \models \forall y \bigwedge_{\bar{b}' \in N} \neg\chi_{\bar{b}'}(\bar{b}, y, \bar{a})$, by (1). For $N \models \neg\delta(\bar{a}')$, note that if $N \models \delta(\bar{a}')$ then $N \models \forall y \bigwedge_{\bar{b}' \in N} \neg\chi_{\bar{b}'}(\bar{b}^*, y, \bar{a}')$ for some $\bar{b}^* \in N$. In particular, $N \models \forall y \neg\chi_{\bar{b}^*}(\bar{b}^*, y, \bar{a}')$, contradicting (2).

So, assuming that (2) of the lemma holds, for each $\mathcal{M} \in \mathcal{K}, \mathcal{N} \in \mathcal{C}_k \setminus \mathcal{K}$, there is an n,r-decomposition formula $\delta_{\mathcal{MN}}$ such that $\mathcal{M} \models \delta_{\mathcal{MN}}$ and $\mathcal{N} \models \neg\delta_{\mathcal{MN}}$. Let

$$\delta = \bigvee_{\mathcal{M} \in \mathcal{K}} \bigwedge_{\mathcal{N} \in \mathcal{C}_k \setminus \mathcal{K}} \delta_{\mathcal{MN}}.$$

This is again effectively a first-order formula, in $\Delta_{n,r}$, because up to equivalence there are only finitely many n,r-decomposition formulas with k free variables. If $\mathcal{M} \in \mathcal{K}$, then $\mathcal{M} \models \bigwedge_{\mathcal{N} \in \mathcal{C}_k \setminus \mathcal{K}} \delta_{\mathcal{MN}}$ and so $\mathcal{M} \models \delta$. If $\mathcal{N} \in \mathcal{C}_k \setminus \mathcal{K}$ and $\mathcal{N} \models \delta$, then $\mathcal{N} \models \bigwedge_{\mathcal{N}' \in \mathcal{C}_k \setminus \mathcal{K}} \delta_{\mathcal{MN}'}$ for some $\mathcal{M} \in \mathcal{K}$. In particular, $\mathcal{N} \models \delta_{\mathcal{MN}}$, contradicting the definition of $\delta_{\mathcal{MN}}$. So $\mathcal{N} \models \neg\delta$. We have shown that $\delta \in \Delta_{n,r}$ defines $\mathcal{K}$. $\square$

Now let $\Delta = \bigcup_{n,r<\omega} \Delta_{n,r}$.

LEMMA 9.4 *The set Δ contains all atomic $\bigcup_{r<\omega} L_r$-formulas and is closed under $\forall, \vee, \neg$, up to equivalence over dedekind-complete time.*

PROOF:

Certainly, any atomic formula is in Δ. Δ is closed under $\vee$, since if $\delta \in \Delta_{n,r}$ and $\delta' \in \Delta_{n',r'}$ then $\delta, \delta' \in \Delta_{\max(n,n'),\max(r,r')}$ and this set is closed under $\vee$.

We check negation now. If $\delta(\bar{z}) \in \Delta_{n,r}$ for some n,r, where $|\bar{z}| = k$, say, then let $\mathcal{K} = \{(M, \bar{a}) \in \mathcal{C}_k : M \models \delta(\bar{a})\}$. If $(M, \bar{a}) \in \mathcal{K}$, $(N, \bar{a}') \in \mathcal{C}_k \setminus \mathcal{K}$, then by lemma 8.2, $\forall$ has a winning strategy in $G_{f(n);n+r}(N, \bar{a}'; M, \bar{a})$ (else $\exists$ has one, so she has a winning strategy in $G_{n;r}(M, \bar{a}; N, \bar{a}')$, contradicting lemma 9.3). So by lemma 9.3, $\mathcal{C}_k \setminus \mathcal{K}$ is $f(n), (n+r)$-definable, so there is $\delta'(\bar{z}) \in \Delta_{f(n),n+r}$ equivalent to $\neg\delta$ in $\mathcal{C}_k$. Hence, Δ is closed under negation, up to equivalence over dedekind-complete time.

For $\exists$, take a formula $\delta = \bigvee_i \bigwedge_j \exists\bar{x}\forall y \chi_{ij}(\bar{x}, y, z, \bar{v}) \in \Delta_{n,r}$, for some n,r. Then

$$\begin{aligned}
\exists z\delta \quad &\equiv \quad \exists z \textstyle\bigvee_i \bigwedge_j \exists\bar{x}\forall y \chi_{ij}(\bar{x}, y, z, \bar{v}) \\
&\equiv \quad \textstyle\bigvee_i \exists z \bigwedge_j \exists\bar{x}\forall y \chi_{ij}(\bar{x}, y, z, \bar{v}) \\
&\equiv \quad \textstyle\bigvee_i \exists z \bigwedge_j \exists\bar{x}_j \forall y \chi_{ij}(\bar{x}_j, y, z, \bar{v}) \\
&\equiv \quad \textstyle\bigvee_i \exists z \exists_j \bar{x}_j \bigwedge_j \forall y \chi_{ij}(\bar{x}_j, y, z, \bar{v}) \\
&\equiv \quad \textstyle\bigvee_i \exists_j z\bar{x}_j \forall y (\bigwedge_j \chi_{ij}(\bar{x}_j, y, z, \bar{v})).
\end{aligned}$$

Here, $\equiv$ denotes logical equivalence; the $\bar{x}_j$ in the third line are fresh variables. But $\bigwedge_j \chi_{ij}(\bar{x}_j, y, z, \bar{v})$ is a quantifier-free L_r-formula, so $\exists z\delta$ is equivalent to a formula of $\Delta_{n',r}$ for some $n' \geq n$. $\square$

Now we prove Kamp's theorem (6.1). By lemma 9.4, any first-order L-formula $\varphi(z)$ is equivalent to a formula $\delta(z) \in \Delta$. That is,

$$\varphi(z) \equiv \bigvee_i \bigwedge_j \exists\bar{x}\, \forall y\, \chi_{ij}(\bar{x}, y, z).$$

But any formula $\exists\bar{x}\, \forall y\, \chi_{ij}(\bar{x}, y, z) \in \Delta_{n,r}$ is readily seen to be equivalent to a temporal formula of rank $n+r+1$. We leave the details as an exercise; see [8, lemma 4.1] for hints.

Talk on relation algebras

For copyright reasons I cannot include the paper [6] here, but it is available at
`http://www.cs.ucl.ac.uk/staff/R.Hirsch/papers/dec.ps`

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